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MATH 3D GEO VR

Learning Materials

Mathematical models for teaching three-dimensional geometry using virtual reality

ENGLISH VERSION

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Learning materials "Mathematical models for teaching three-dimensional geometry using virtual reality"

Created by the Math3DgeoVR consortium.

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1. TOPIC: Systems of linear equations

1. Justification for topic choice

In mathematics, a system of linear equations (or linear system) is a collection of one or more linear equations involving the same variables. The module describes the problems of solving systems of linear equations with real coefficients and at the same time refers to their geometrical interpretation.

Linear systems theory is the basis and fundamental part of linear algebra, a subject used in most branches of modern mathematics. Computational algorithms for finding solutions are an important part of numerical linear algebra and play a significant role in engineering, physics, chemistry, computer science and economics. Moreover - a system of non-linear equations can often be approximated by a linear system (see linearization), which is a helpful technique when creating a mathematical model or computer simulation of even complex systems.

The module not only supports finding a solution to a system of equations but also allows the user to associate an algebraic expression with its geometric interpretation. It can also be a tool that can be used to solve problems requiring the use of systems of linear equations. It is rare to find equations that accurately model the problem. Rather, it is likely that the learner will encounter a situation in which he knows the key information and needs to adapt the system of equations, or knows the solution and needs to modify the coefficients of the system. Thus, the module also allows the user to learn the basics of the analytical solution to the task. For example:

Given a situation that represents a system of linear equations, write the system of equations and identify the solution.

a) Identify unknown quantities in a problem and represent them with variables.

b) Write a system of equations that models the problem’s conditions.

c) Solve the system.

d) Check the proposed solution.

The standard algorithm for solving a system of linear equations is based on the so-called Gaussian elimination with some modifications.

2. Historical background

Around 4000 years ago, the people of Babylon knew how to solve a simple 2x2 system of linear equations with two unknowns. Around 200 BC, the Chinese published "Nine Chapters of the Mathematical Art". They displayed the ability to solve a 3x3 system of equations (Perotti: Retrieved from http://www.science.unitn.it/~perotti/History of Linear Algebra.pdf). The power and progress of linear algebra did not come to fruition until the late 17th century. Then the subject of determinants and values related to a square matrix, studied by the creator of calculus

Leibnitz, emerged. At the turn of the 20th century, Gauss introduced a procedure for solving a system of linear equations. His work mainly dealt with linear equations and had not yet introduced the idea of matrices or their notation. He dealt with equations of various numbers and variables, and analysed the work of Euler, Leibnitz, and Cramer before the 19th century. The term "matrix" was introduced in 1848 by J.J. Sylvester. The foundations of matrix algebra theory come from the work of Arthur Cayley in 1855.

Although linear algebra is a fairly new subject when compared to other mathematical practices, its uses are widespread. With the efforts of calculus-savvy Leibnitz, the concept of using systems of linear equations to solve unknowns was formalized. Other efforts from scholars like Cayley, Euler, Sylvester, and others changed linear systems into the use of matrices to represent them. Regardless of the technology, Gaussian elimination still proves to be the best way to solve a system of linear equations.

Even as scientists update their textbooks, the basics remain the same.

3. Learning outcomes

On completion this module students should be able to understand and correctly select the theoretical knowledge that will allow him/her to achieve:

- Skill to define the meaning of a system of linear equations with two and three variables in linear algebra notation.
- Skill to give examples along with their geometrical interpretation.
- Ability to choose the correct mathematical model for a task using linear equations.
- Ability to solve problems of a system of linear equations by various methods, e.g. substitution, elimination and combination.

Students who can understand mathematical concepts can explain the concepts they have learned, distinguish which are examples and which are not examples based on the definitions and materials provided, and apply the concepts they have learned in solving related problems. This means that the student has the ability to:

- Determine the properties that need to be solved (e.g. the number of equations, whether they are homogeneous).
- Declare the existence of a solution (e.g. rank of a matrix, determinant).
- Correctly choose the method of solving the given system.
- Determine geometric representation (e.g. as support when using a solution method).

4. Theoretical foundations

Practical problems in many fields of study such as - biology, business, chemistry, computer science, economics, electronics, engineering, physics and the social sciences - can often be reduced to solving a system of linear equations. Linear algebra arose from attempts to find systematic methods for solving these systems.

For example - if a, b and c are real numbers, the graph of an equation of the form:

$$a\,x + b = c$$

is a straight line (if a and b are not both zero), so such an equation is called a linear equation in the variable x.

However, it is often convenient to write the variables as $x_1, x_2, \ldots, x_n$, particularly when more than two variables are involved. An equation of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is called a linear equation in the n variables $x_1, x_2, \ldots, x_n$. Here $a_1, a_2, \ldots, a_n$ denote real numbers (called the coefficients of $x_1, x_2, \ldots, x_n$, respectively) and b is also a number (called the constant term of the equation). A finite collection of linear equations with the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations of these variables.

Given a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, a sequence $s_1, s_2, \ldots, s_n$ of n numbers is called a solution to the equation if:

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b,$$

that is, if the equation is satisfied when all x's are substituted with corresponding s's $(x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n)$. A sequence of numbers is called a solution to a system of equations if it is a solution to every equation in the system.

Definition 1.1. The general $m \times n$ system of linear equations is of the form:

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + \cdots a_{1n}\,x_n &= b_1, \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots a_{2n}\,x_n &= b_2, \\ &\vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots a_{mn}\,x_n &= b_m, \end{aligned}$$

where the system coefficients $a_ij,\ i = 1,2, \ldots, m, j = 1,2, \ldots, n$ and the system constants b_j are given scalars and $x_1, x_2, \ldots, x_n$ denote the unknowns in the system. If $b_i = 0$ for all i, then the system is called homogeneous; otherwise, it is called nonhomogeneous.

Definition 1.2. By a solution to the system of linear equations we mean an ordered n −tuple of scalars, $(c_1, c_2, \ldots, c_n)$, which, when substituted for $x_1, x_2, \ldots, x_n$ into the left side of system, yield the values on the right side. The set of all solutions to the system is called the solution set to the system.

Definition 1.3. A system of equations that has at least one solution is named consistent, whereas a system that has no solution is called inconsistent.

In the case of equations with an infinite number of solutions, variables should be introduced, which are called parameters. The set of solutions described in this way has a parametric form and is called the general solution of the system. It turns out that solutions to any system of equations (if there are solutions) can be given in the parametric form (i.e. variables x_1, x_2, ...are given in the form of new independent variables s, t, etc.).

Our problem will be to determine whether a given system is consistent and then, if it is, to find its solution set.

Definition 1.4. Naturally associated with the system of linear equations there are the following two matrices:

- *The matrix of coefficients* $A = [a_{11} \cdots a_1 n \vdots\ddots\vdots\ a_m 1 \cdots a_m n]$.
- *The augmented matrix* $[A|B] = [a_{11} \cdots a_1 n \vdots\ddots\vdots\ a_m 1 \cdots a_m n | b_1 \dots b_m]$.

The augmented matrix completely characterizes a system of equations since it contains all of the system coefficients and system constants. Note that the coefficient matrix is a matrix consisting of the first n columns of B.

Definition 1.5. The row rank r_r of a matrix is the maximum number of rows, thought of as vectors, which are linearly independent. Similarly, the column rank r_c is the maximum number of columns which are linearly independent. It is an important result, not too hard to show that the row and column ranks of a matrix are equal to each other. Thus, one simply speaks of the rank of a matrix $r = r_r = r_c$.

Theorem 1.1 *(Kronecker - Cappelli). A system of linear equations has a solution if and only if:* $r(A) = r([A|B])$.

- ## Elementary Row operations

The elementary row operations consist of the following:

- Interchange two rows.
- Multiply a row by a nonzero number.
- Replace a row with any multiple of another row added to it.

Geometric interpretation of linear systems

The geometric interpretation of a linear system of two equations in two unknowns is the geometric presentation of the two straight lines that correspond to the linear functions of these equations. Typically, textbooks contain schematic diagrams of the following type:

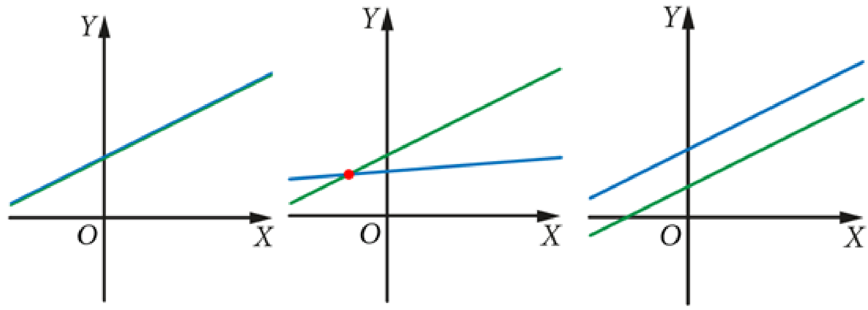

Figure 1.1.

A single equation in three unknowns can be interpreted geometrically in 3-dimensional space. An equation:

$$Ax + By + Cz = D$$

has solutions that form a plane as long as at least one of A, B, C is nonzero. If we look at a system of such linear equations, the solution set is the set of all points lying in all the corresponding planes.

Typically, textbooks contain schematic diagrams of the following type:

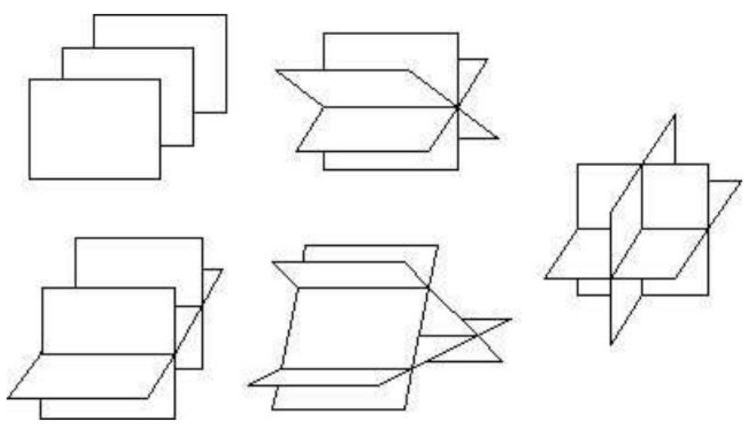

Figure 1.2. Interaction of lines and planes by Dan Sunday, http://geomalgorithms.com

Two linear systems using the same set of variables are equivalent, if each of the equations in the second system can be derived algebraically from the equations in the first system, and vice versa. Two systems are equivalent if either both are inconsistent or each equation of each of them is a linear combination of the equations of the other one. It follows that two linear systems are equivalent if and only if they have the same solution set.

Understanding mathematical concepts is a very important skill obtained by students while learning mathematics. Returning to the general discussion of the system of linear equations, we will consider some fundamental questions:

1. Does the system have a solution?
2. If the answer to the first question is yes, then how many solutions are there?
3. How do we determine all the solutions?

5. Applications to everyday life

To link systems of linear equations with the practical application of their geometrical interpretation, in the module in the application section, an issue was selected related to the selection of energy beams. Thanks to this, the user not only modifies the system of equations, but also has to observe changes related to the visualization of the applied algebraic changes. There are a lot of algorithms for reconstructing a Computed Tomography (CT) image of a medium. We focused on the linear algebraic methods.

In some cases, plain X-ray examination from one direction is not good enough and vital information could go undetected (such as a tumour behind a bone). By using CT instead, the possibility of missing this information will be significantly reduced since CT uses X-ray beams from many different directions to create a cross-sectional image of the medium, this way objects within the medium cannot cover each other.

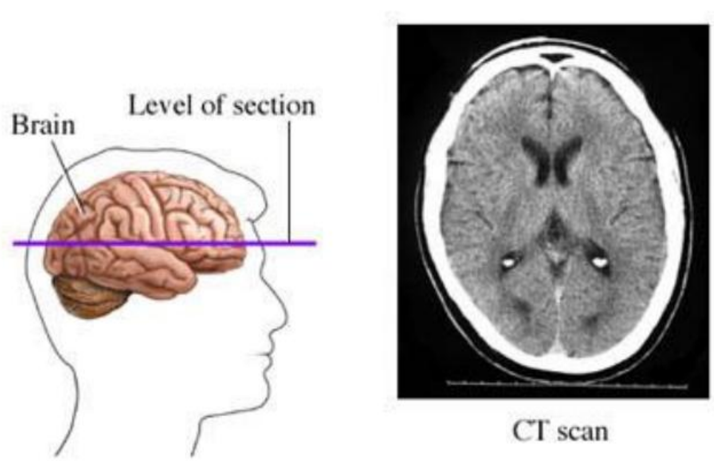

Figure 1.3. CT slice. Source: https://dspace.bracu.ac.bd/xmlui/bitstream /handle/10361/10915/13216001_MNS.pdf?sequence=1&isAllowed=y

A slice of the brain has been superimposed by a grid for simplicity and to allocate objects grid-wise. That is, each and every grid contains an object to be scanned, analysed and identified.

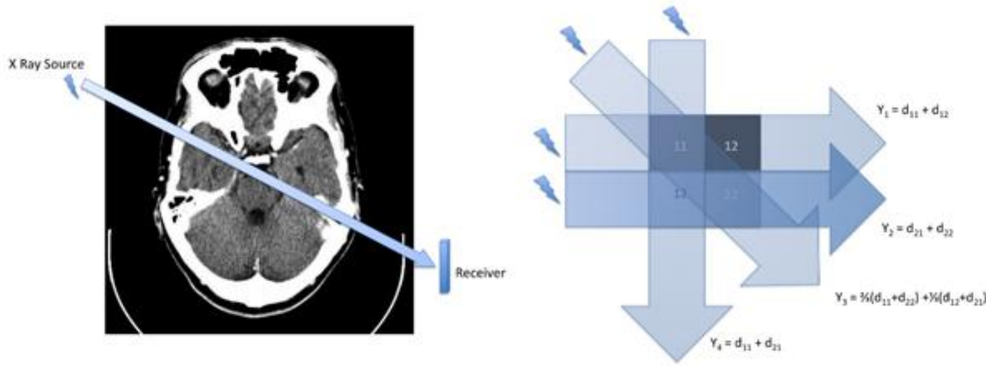

Figure 1.4. A single slice obtained by a CT scan. Each slice is a reconstructed image obtained by recording the attenuation of X-rays through the tissues along a vast number of directions. The picture shows the X-ray source and the corresponding receiver used to measure attenuation along a specific direction.
On the right - linear relationship between the observed intensity ratios.
Source: *https://ecampusontario.pressbooks.pub/linearalgebraandapplications/chapter/motivating-example/*

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2. TOPIC: Partial derivatives

1. Justification for topic choice

The derivative of a single-variable function $f(x)$ tells us how much $f(x)$ changes as x changes. There is no ambiguity when we speak about the rate of change of $f(x)$ with respect to x since x must be constrained to move along the x-axis. The situation becomes more complicated, however, when we study the rate of change of a function of two or more variables. The obvious analogue for a function of two variables $f(x, y)$ would be something that tells us how quickly $f(x, y)$ increases as x and y increase. However, in most cases this will depend on how quickly x and y are changing relative to each other. For multivariable functions, their values will change when one or more of the input values change, that's why it is important to calculate the change in the function itself. This can be investigated by holding all but one of the variables constant and finding the rate of change of the function with respect to the one remaining variable. Knowing how to calculate partial derivatives allows one to study and understand the behavior of multivariable functions. This process is called partial differentiation. This process is called partial differentiation. This opens a wide range of applications in calculus such as the tangent planes, ...

2. Historical background

One of the first known uses of this symbol ∂ in mathematics is by Marquis de Condorcet from 1770, who used it for partial differences. The modern partial derivative notation was created by Adrien-Marie Legendre (1786), although he later abandoned it; Carl Gustav Jacob Jacobi reintroduced the symbol in 1841.

3. Learning outcomes

On completion this module students should be able to

- definition and computation techniques for first and higher partials, both at a specific point
- understanding the concept of partial differentation
- derivative of a function partially with respect to each of its variables in turn
- evaluation of the first partial derivatives
- formulation of the second partial derivatives

Prerequisites: Before starting this module students should

- know the principle of differentiating a function of one variable

4. Theoretical foundations

- ## The partial derivative of function $f(x, y)$ with respect to x.

In the relation $z = f(x, y)$ the independent variables are x and y and the dependent variable z. From the geometric interpretation of a function of two variables, we know that with each change in the value of x or y, we obtain a sequence of curves, each of which lies in a different plane and each of which is part of the surface we would like to sketch. Now both of the variables x and y may change simultaneously inducing a change in z. However, rather than consider this general situation, to begin with we shall hold one of the independent variables fixed. This is equivalent to moving along a curve obtained by intersecting the surface by one of the coordinate planes. By keeping y as a constant and varying x only, z becomes a function of x alone.

Definition 2.1. Suppose $z = f(x, y)$ is a two-variable function. Then, the first partial derivative of $f(x, y)$ with respect to x at $[x_0, y_0]$ is given by $\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$

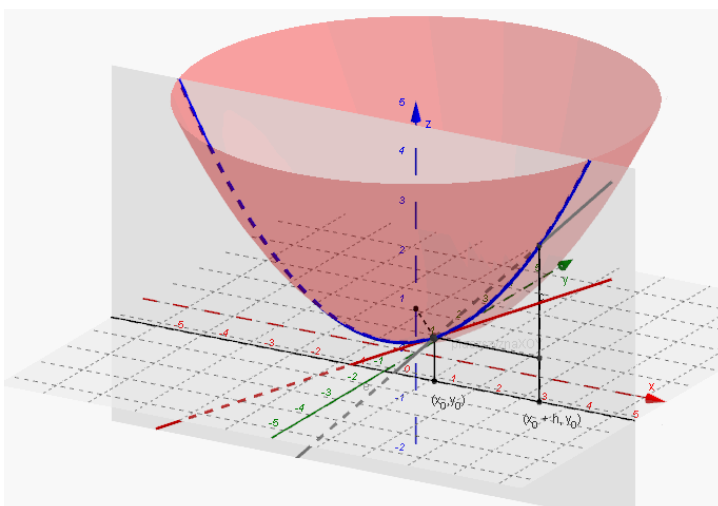

Figure 2.1.

The derivative of function z with respect to x (y is kept constant) is called the partial derivative of z with respect to x and is denoted by $\frac{dz}{dx}, \frac{df}{dx}, f'_x$ or $D_x f$.

Graphically, $\frac{df(x,y)}{dx}$ tell us the instantaneous rate of change of the function if we hold y fixed and move parallel to the x-axis in the positive direction.

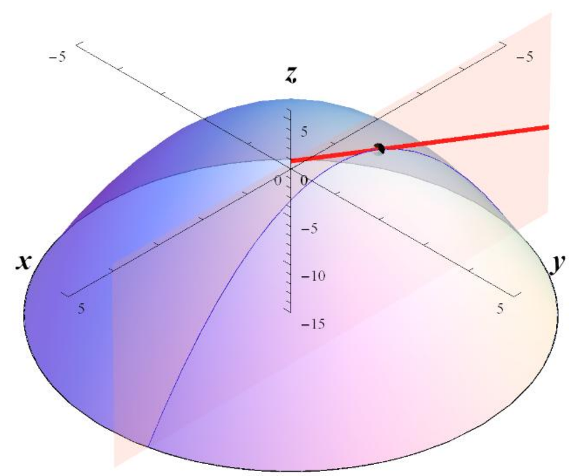

Figure 2.2.

Figure 2.2 shows as we move in the $+x$-direction from $(-1,1)$, $\frac{df(x,y)}{dx}(-1,1)$ is positive, so $f(x,y)$ is increasing in that direction.

- ### The partial derivative of function $f(x,y)$ with respect to y.

Analogously by keeping x as a constant and varying y only, z becomes a function of y alone. Now we can define:

Definition 2.2. Suppose $z = f(x,y)$ is a two-variable function. Then, the first partial derivative of $f(x,y)$ with respect to y at $[x_0, y_0]$ is given by $\frac{\partial f(x,y)}{\partial y} = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$
provide the limit exists.

Similarly, the derivative of function z with respect to y (x is kept constant) is called the partial derivative of z with respect to y and is denoted by $\frac{dz}{dy}, \frac{df}{dy}, f_y'$ or $D_y f$.

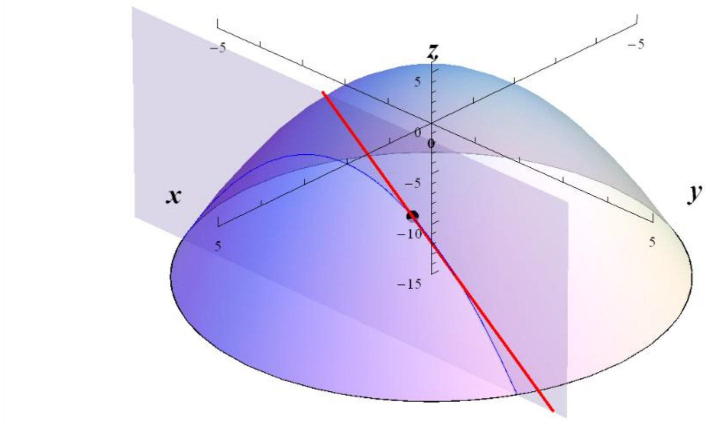

Figure 2.3.

Figure 2.3 shows as we move in the $+y$-direction from $\left(2,\frac{2}{3}\right)$, $\frac{df(x,y)}{dx}\left(2,\frac{2}{3}\right)$ is negative, so $f(x,y)$ is increasing in that direction.

Graphically, $\frac{df(x,y)}{dy}$ tell us the instantaneous rate of change of the function if we hold x fixed and move parallel to the y-axis in the positive direction.

Remark 2.1. In multivariable calculus, we use the symbol ∂ (this swirly-d symbol, ∂, often called "del", is used to distinguish partial derivatives from ordinary single-variable derivatives, where we use the symbol d).

To evaluate a partial derivative of the function $f(x,y)$ with respect to x, we need only pretend that all the other variables (i. e., everything except x) that $f(x,y)$ depends on are constants, and then just evaluate the derivative of $f(x,y)$ with respect x as a normal one-variable derivative.

All of the derivative rules (the product rule, quotient rule, chain rule, etc.) from one-variable calculus still hold: there will just be extra variables floating around.

One possible misconception is that the partial derivative with respect to a particular variable depends only on that variable. That is not true. The expression for the partial derivative with respect to x potentially depends on both x and y. What this means that the value of partial derivative depends on the location of the point, even the other coordinate.

The exception is the case of additively separable functions. In other words, if we can write $F(x,y)$ as $f(x)+g(y)$ where $f(x)$ is a function of one variable x and $g(y)$ is a function of one variable y. Then $\frac{\partial F(x,y)}{\partial x}=f'^{(x)}$ and is independent of y and $\frac{\partial F(x,y)}{\partial y}=g'^{(y)}$ and is independent of variable x.

We can generalize partial derivatives to functions of more than two variables: for each input variable, we get a partial derivative with respect to that variable. The procedure remains the same: treat all variables except the variable of interest as constants, and then differentiate with respect to the variable of interest.

Geometrically, the partial derivatives can be interpreted as slopes of tangent lines to graphs of functions of the one variable being differentiated with respect to, once we fix the value of the other variable.

The partial derivative f'_x captures how fast the function $f(x, y)$ is changing in the x-direction, and f'_y captures how fast the function $f(x, y)$ is changing in the y-direction.

In general, if $z = f(x, y)$ is a function of more than two independent variables, then the partial derivative of $z = f(x, y)$ with respect to any one of the variables, keeping all other variables constant, is the partial derivative of z with respect to that variable.

- ## Higher- order partial derivatives of function of more variables

Like in the one variable case, we also have higher-order partial derivatives by iterating partial differentiation. In general, the first order partial derivatives $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ can be differentiated successively to the partial derivatives of higher order.

For instance $\frac{\partial f^2(x,y)}{\partial x^2}$ is the derivative of $\frac{\partial f(x,y)}{\partial x}$ with respect to x. For a function of two variables x and y, we have four second order partial: $\frac{\partial f^2(x,y)}{\partial x^2}, \frac{\partial f^2(x,y)}{\partial x\,\partial y}, \frac{\partial f^2(x,y)}{\partial y\,\partial x}, \frac{\partial f^2(x,y)}{\partial y^2}$.

Remark 2.1. Partial derivatives in subscript notation are applied left-to-right, while partial derivatives in differential operator notation are applied right-to-left. In practice, the order of partial derivatives rarely matters.

Theorem 2.2. *If both partial derivatives* $\frac{\partial f(x,y)}{\partial x}$ *and* $\frac{\partial f(x,y)}{\partial y}$ *are continuous, then they are equal.*

Remark 2.3.

- In other words, this mixed partial derivatives are always equal (given mild assumptons about continuity), so there are really only three second-order partial derivatives.
- This theorem can be proven using the limit definition of derivative and Mean Value Theorem. The details of proof are unenlightening, so we will omit them.
- We can continue on and take higher-order partial derivatives than second-order.

5. Applications to everyday life

Tangents have many applications in everyday life, from architecture to engineering to physics. We say that a tangent is a line that touches the curve at an external point. In geometry, a tangent is a straight line that touches a curve at a single point.

The tangent plane represents the surface that contains all tangent lines of the curve at a point $P_0 = [x_0, y_0, z_0]$, that lies on the surface and passes through this point. So, the tangent plane allows us to predict the behaviors of surfaces at certain points of the function.

Definition 2.3. Let $P_0 = [x_0, y_0, z_0]$ be a point on a surface S, and let C be a curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves at P_0 lie in the same plane, then this plane is called the tangent plane to S at point P_0.

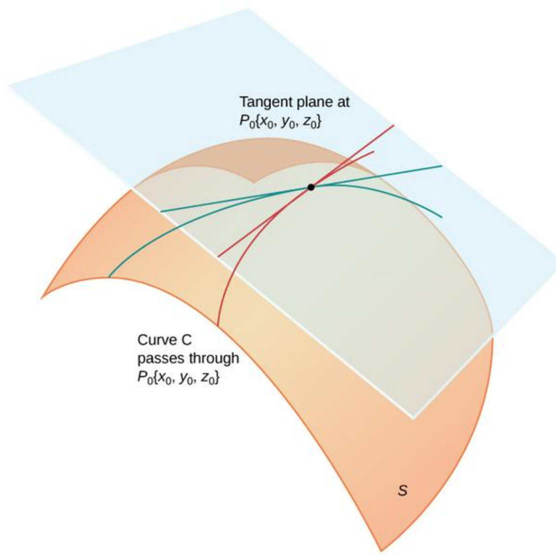

Figure 2.4 The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S that pass through P_0

Definition 2.4. Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = [x_0, y_0]$ be a point in the domain of $f(x, y)$. Then the equation of the tangent plane to S at point P_0 is given by $z = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)$

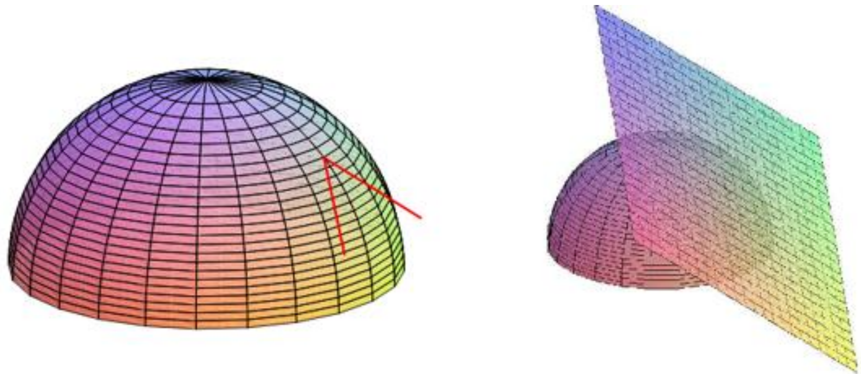

Figure 2.5. Tangent vectors and tangent plane

Tangents have several important applications across various fields, particularly in mathematics, physics, engineering, and geometry. Some applications of tangents are:

- Geometry and Trigonometry
- Architecture
- Engineering and Design
- Physics

Geometry and Trigonometry

Tangents play an important role in the study of curves. They are lines that touch a curve at a single point, perpendicular to the curve's radius at that point. Tangents are used to define geometric properties such as the radius, diameter, chord, and arc length of circles.

Architecture

Tangents are used to design curves in buildings, such as arches and domes. By using tangents, Architects use tangents to create curves that are pleasing to the eye.

Engineering and Design

Tangents are used to design curves in roads, bridges, and other structures. By using tangents, engineers can create curves that are safe and efficient for travel. Curves are used in automotive design, aerospace engineering, and civil engineering to design roads, bridges, and other structures with smooth transitions and efficient curves.

Physics

Tangents are used to analyze the motion of objects, for example, the tangent to a projectile's trajectory at any point represents the direction of the projectile's velocity at that point.

- Tangent in Day-to-Day Life Situations
    - Designing a Parabolic Arch - A parabolic arch is a curve that is shaped like a parabola. Parabolas are often used in architecture to create arches and domes.
    - Designing a Circular Bridge - A circular bridge is a bridge that is shaped like a circle. Circular bridges are often used to span rivers and other obstacles.
    - Designing a Roller Coaster - Roller coasters are designed to provide riders with a thrilling experience. To design a roller coaster, engineers use tangents to determine the shape of the track. The tangents ensure that the roller coaster is safe and provides riders with a smooth ride.
    - Designing a Car Race Track - Car race tracks are designed to provide drivers with a challenging and exciting experience. To design a car race track, engineers use tangents to determine the shape of the track. The tangents ensure that the track is safe and provides drivers with a fair race.
    - Designing Golf Course - Golf courses are designed to provide golfers with a challenging and enjoyable experience. To design a golf course, architects use tangents to determine the shape of the fairways and greens. The tangents ensure that the
    - Designing Skateboard Park - Skateboard parks are designed to provide skateboarders with a safe and fun place to practice their skills. To design a skateboard park, architects use tangents to determine the shape of the ramps and bowls. The tangents ensure that the skateboard park is safe and provides skateboarders with a variety of challenges.

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3. TOPIC: Gradient of a scalar field

1. Justification for topic choice

A vector field or a scalar field can be differentiated with respect to position in three ways to produce another vector field or scalar field. There are three derivatives: the gradient of a scalar field, the divergence of a vector field and the curl of a vector field.

This module studies the gradient of a scalar field.

2. Historical background

The gradient was denoted Δ by Hamilton in 1846. By 1870 it was denoted ∇, an upside-down delta, and therefore called "atle". In 1871 Maxwell wrote, "The quantity ∇P is a vector. I venture, with much diffidence, to call it the slope of P." The name "slope" is no longer used, having been replaced by "gradient". "Gradient" goes back to the word "grade", the slope of a road or surface. The name "del" first appeared in print in 1901, in *Vector Analysis, A text-book for the use of students of mathematics and physics founded upon the lectures of J. Willard Gibbs,* by E.B. Wilson.

3. Learning outcomes

On completion this module students should be able to

- find the gradient of a scalar field
- to compute the directional derivative

Prerequisites: Before starting this module students should

- be familiar with the concept of a function of two or three variables
- be familiar with the concept of partial differentiation
- be familiar with scalar and vector fields

4. Theoretical foundations

- ### Gradient

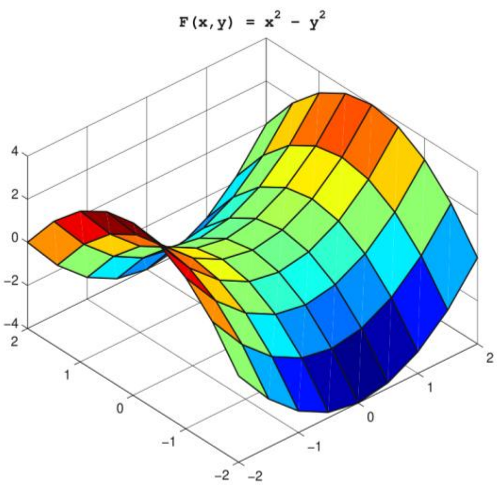
Definition 3.1. The gradient of the scalar field $f(x, y, z)$ is a vector grad$f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$, where $\vec{i}, \vec{j}, \vec{k}$ are the standard unit vectors in the directions of x, y, z coordinates, respectively.

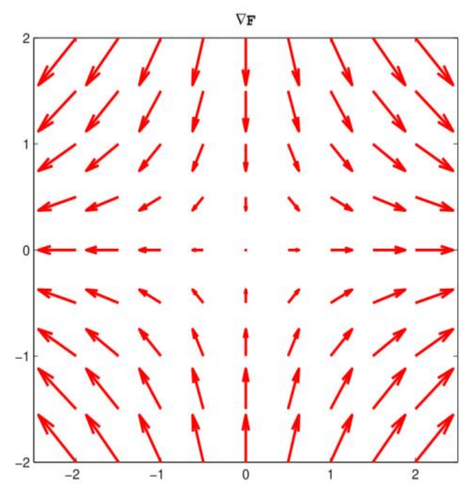
In some applications it is customary to represent the gradient as a row vector or column vector of its components in a rectangular coordinate system. Often, instead of grad$f$, the notation ∇f is used. (∇ is a vector differential operator called "del" or "nabla" defined by $\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$. As a vector differential operator, it retains the characteristics of a vector while also carrying out differentiation.)

If f is a function of some other number of variables, generally n, the gradient is defined analogously: the component of the vector in any coordinate x_i, $i = 1,2,\dots,n$, is the partial derivative $\frac{\partial f}{\partial x_i}$ of f with respect to x_i.

Function f is a scalar function (scalar field) but grad$f$, is a vector-valued function (vector field): it takes the same number of arguments as f does, and outputs a vector in the same number of coordinates.

In the next figures there are several representations of scalar functions and corresponding gradient vector fields. First one is the function $f(x,y) = x^2 - y^2$, in the next figure there is the gradient of the function $f(x,y) = -(\cos^2 x + \cos^2 y)^2$ depicted as a projected vector field on the bottom plane and the gradient, represented by the blue arrows, denotes the direction of greatest change of a scalar function.





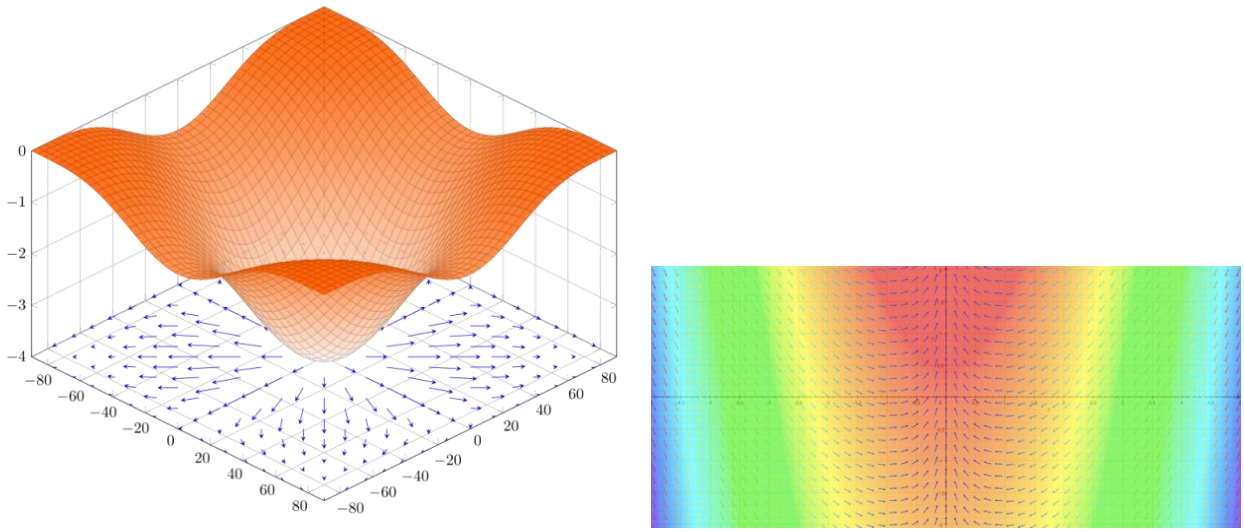


Figure 3.1.

In the next figure there is gradient depicted on the surface - red arrows represent the greatest increase, blue slower, and on the top increase and gradient are zero.

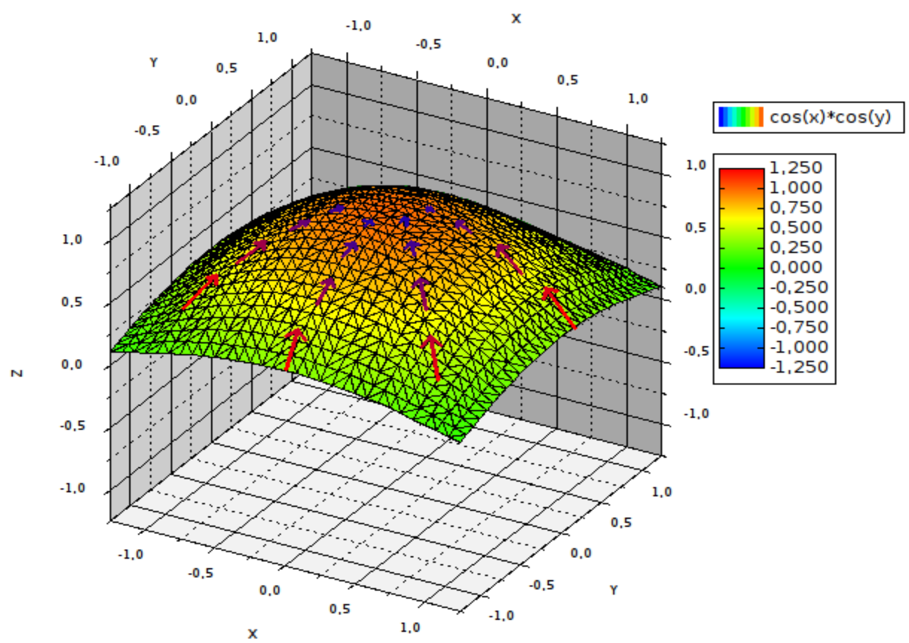


Figure 3.2.

Properties:

- **Linearity:** The gradient is linear in the sense that if $f_i,\ i = 1,2,\dots,n$ are real-valued differentiable functions, and $c_i \in \mathbb{R}, i = 1,2,\dots,n$ are constants, then,

$$\text{grad}(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n) = c_1 \text{grad} f_1 + c_2 \text{grad} f_2 + \cdots + c_n \text{grad} f_n.$$

- **Product rule:** If f and g are real-valued differentiable functions, then the product rule asserts that the product $f\, g$ is differentiable, and

$$\text{grad}(f\, g) = f\, \text{grad} g + g\, \text{grad} f.$$

The vector grad$f$ gives the magnitude and direction of the greatest rate of change of f at any point A. The gradient vector $\nabla f(x_0, y_0)$ is orthogonal (or perpendicular) to the level curve $f(x,y) = k$ at the point $A = [x_0, y_0]$. Likewise, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x,y,z) = k$ at the point $A = [x_0, y_0, z_0]$. The following figure shows the positive direction of the gradient vector at different points of the contours of function f_2. The direction of the positive gradient is indicated by the red arrow. The tangent line to a contour is shown in green.

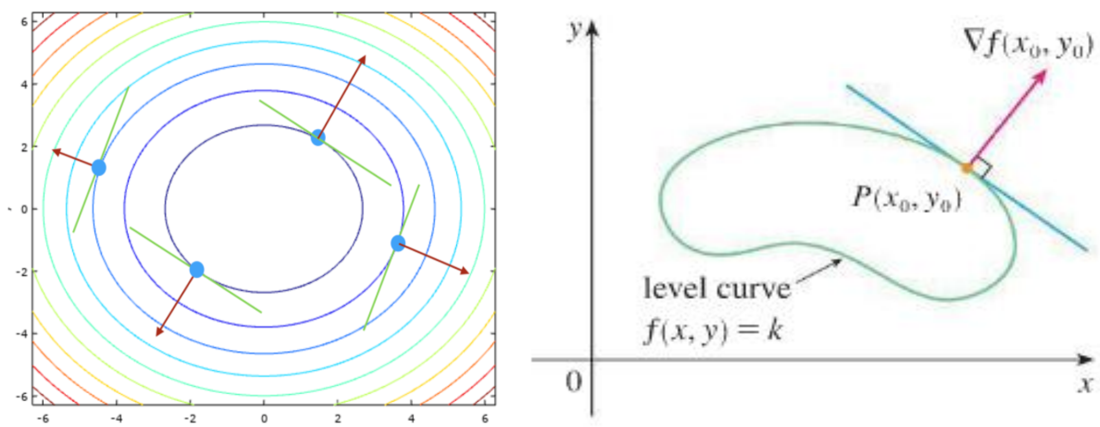


Figure 3.3.

In the next figures there is the function $f(x,y) = sin(x + y^2)$ and its contour lines.

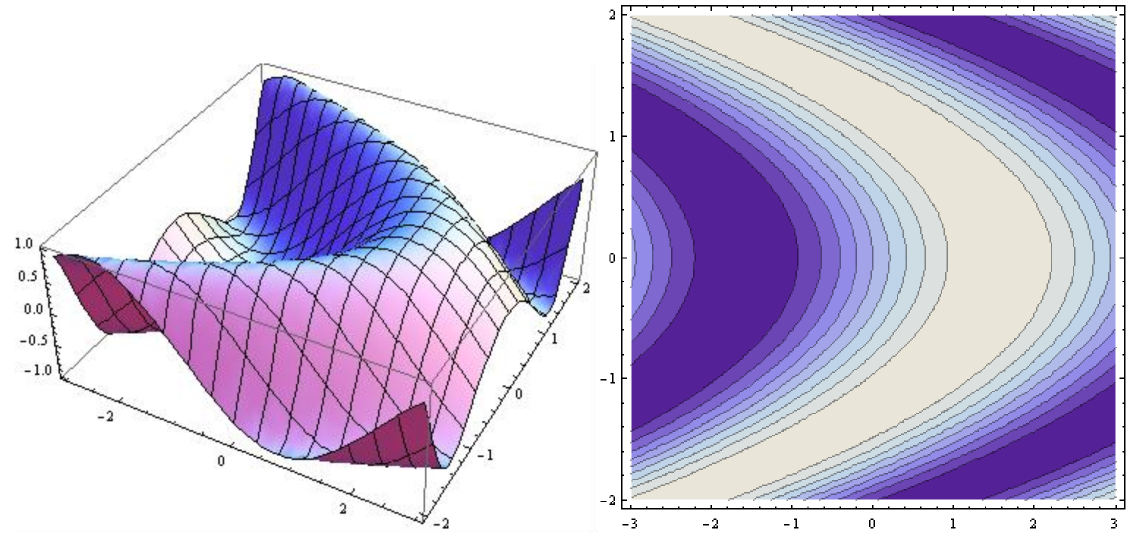


Figure 3.4.

In the next figure there are visualized equipotential surfaces of function

$$f(x,y,z) = \sin(x + y^2) + z.$$

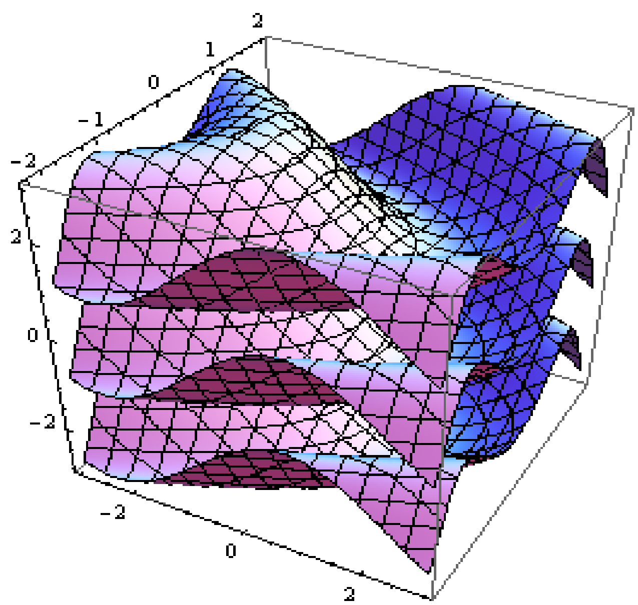


Figure 3.5.

Since gradient vector is $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, the tangent plane to the surface given by $f(x,y,z) = k$ at $A = [x_0, y_0, z_0]$ has the equation

$$\frac{\partial f(A)}{\partial x}(x - x_0) + \frac{\partial f(A)}{\partial y}(y - y_0) + \frac{\partial f(A)}{\partial z}(z - z_0) = 0.$$

When we need a line that is orthogonal to a surface at a point - the normal line, this is easy enough to get if we recall that the equation of a line only requires that we have a point and a parallel vector. Since we want a line that is at the point $A = [x_0, y_0, z_0]$ we know that this point must also be on the line and we know that $\nabla f(x_0, y_0, z_0)$ is a vector that is normal to the surface and hence will be parallel to the line. Therefore, the equation of the normal line is,

$$\begin{aligned} x &= x_0 + t\,\frac{\partial f(x_0, y_0, z_0)}{\partial x} \\ y &= y_0 + t\,\frac{\partial f(x_0, y_0, z_0)}{\partial y} \\ z &= z_0 + t\,\frac{\partial f(x_0, y_0, z_0)}{\partial z},\ t \in R \end{aligned}$$

In previous we introduced the gradient in Cartesian coordinates. The gradient could also be expressed in other coordinates, e.g., polar, cylindrical and spherical coordinates. In polar coordinates the gradient is given by

$$\text{grad} f(r, \varphi) = \nabla f = \frac{\partial f}{\partial r}\,\overrightarrow{e_r} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\,\overrightarrow{e_\varphi},$$

in cylindrical coordinates the gradient is given by

$$\text{grad} f(r, \varphi, z) = \nabla f = \frac{\partial f}{\partial r}\,\overrightarrow{e_r} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\,\overrightarrow{e_\varphi} + \frac{\partial f}{\partial z}\,\overrightarrow{e_z},$$

where r is the axial distance, φ is the azimuthal or azimuth angle, z is the axial coordinate, and $\overrightarrow{e_r}, \overrightarrow{e_\varphi}, \overrightarrow{e_z}$ are unit vectors pointing along the coordinate directions.

In spherical coordinates the gradient is given by

$$\text{grad} f(r, \varphi, \theta) = \nabla f = \frac{\partial f}{\partial r}\,\overrightarrow{e_r} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\,\overrightarrow{e_\varphi} + \frac{1}{r\sin\varphi}\frac{\partial f}{\partial \theta}\,\overrightarrow{e_\theta},$$

where r is the radial distance, φ is the azimuthal angle and θ is the polar angle, and $\overrightarrow{e_r}, \overrightarrow{e_\varphi}, \overrightarrow{e_\theta}$ are again local unit vectors pointing in the coordinate directions.

- ## Directional derivative

The change in a function f in a given direction (specified as a unit vector $\vec{l}$) is determined from the scalar product $\text{grad} f \cdot \vec{l}$. This scalar quantity is called the directional derivative.

Definition 3.2. If $\vec{l} = (l_x, l_y)$ is a unit vector, then the directional derivative of f in the direction of $\vec{l}$ at any point $A = [x_0, y_0]$, denoted $\frac{df(A)}{d\vec{l}}$, is defined to be the limit $\frac{df(A)}{d\vec{l}} = \lim_{h\to 0} \frac{f(x_0+hl_x, y_0+hl_y)-f(x_0,y_0)}{h}$, provided that the limit exists.

The corresponding definition when f has more variables is analogous. When $\vec{l}$ is the unit vector in one of the coordinate directions, the directional derivative reduces to the corresponding partial derivative.

Theorem 3.1. *If $\vec{l}$ is any unit vector, and f is a function all of whose partial derivatives are continuous, then the directional derivative satisfies $\frac{df}{d\vec{l}} = gradf \cdot \vec{l}$.*

This result requires the direction $\vec{l}$ to be a unit vector. If the desired direction is not a unit vector, it is necessary to normalise the direction vector first.

Sometimes we will give the direction of changing x and y as an angle. For instance, we may say that we want the rate of change of f in the direction of α. The unit vector that points in this direction is given by $\vec{l} = (\cos\alpha, \sin\alpha)$. In three dimensional case $\vec{l} = (\cos\alpha, \cos\beta, \cos\gamma)$, where α, β, γ are angles that vector $\vec{l}$ makes with the coordinate axes, there are called the direction cosines. In such case the directional derivative is computed as

$$\frac{\mathrm{d}f df(x,y)}{d\vec{l}} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha\,, \qquad \frac{df(x,y,z)}{d\vec{l}} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma.$$

From the gradient theorem for computing directional derivatives, we can deduce several corollaries about how the magnitude of the directional derivative depends on the direction $\vec{l}$:

Corollaries: Suppose f is a differentiable function with gradient grad$f$ and $\vec{l}$ is a unit vector. Then the following hold:

1) The maximum value of $\frac{df}{d\vec{l}}$ occurs when $\vec{l}$ is a unit vector pointing in the direction of grad$f$, if $gradf \neq 0$, and the maximum value is $+|$grad$f|$. In other words, the gradient points in the direction where f is increasing the most rapidly.
2) The minimum value of $\frac{df}{d\vec{l}}$ occurs when $\vec{l}$ is a unit vector pointing the opposite direction of grad$f$, if grad$f \neq 0$, and the minimum value is $-|$grad$f|$. In other words, the gradient points in the opposite direction from where f is decreasing the most rapidly.
3) The value of $\frac{df}{d\vec{l}}$ is zero if and only if $\vec{l}$ is orthogonal to the grad$f$.

5. Applications to everyday life

Consider a room where the temperature is given by a scalar field, T, so at each point $[x, y, z]$ the temperature is $T(x, y, z)$, independent of time. At each point in the room, the gradient of T at that point will show the direction in which the temperature rises most quickly, moving away from $[x, y, z]$. The magnitude of the gradient will determine how fast the temperature rises in that direction.

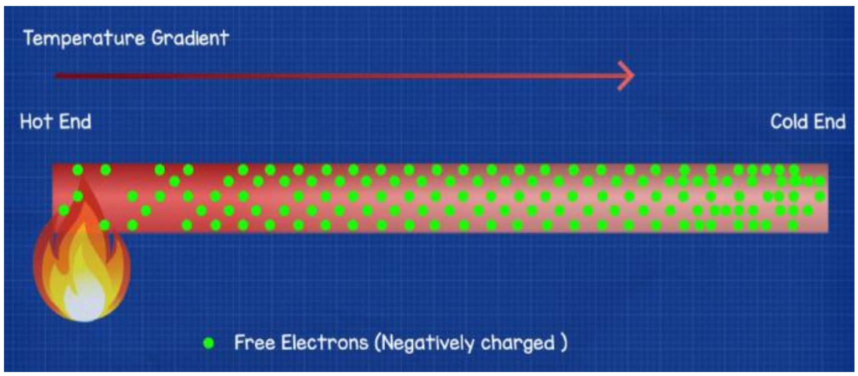


Figure 3.6.

In the next figure there are temperature contours and heat flow lines for a metal plate. The direction of the heat flow is along the flow lines which are orthogonal to the contours (the dashed lines in figure (b)); this heat flow is proportional to the vector field grad$f$.

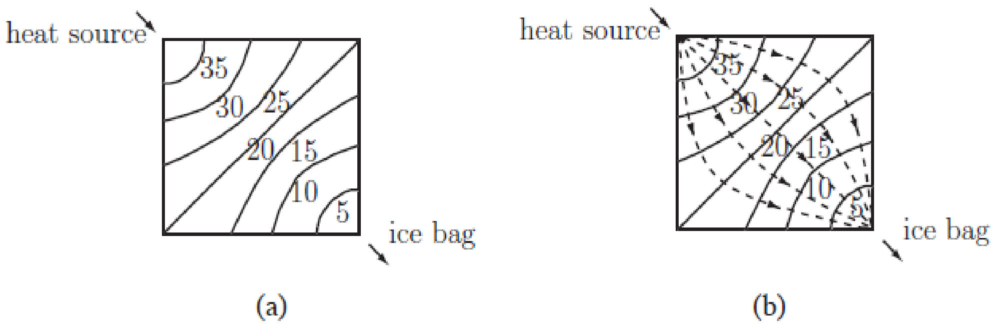


Figure 3.7.

Also many times there are published daily maps showing the temperature throughout the nation with the aid of contour lines. Next figure is an example

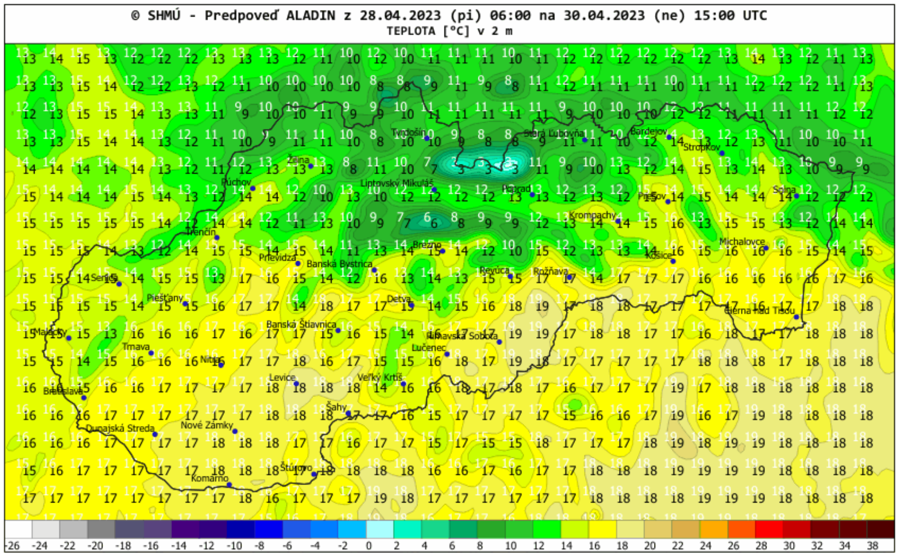


Figure 3.8.

Now consider a surface whose height above sea level at point $[x, y]$ is $f(x, y)$. The gradient of f at a point is a plane vector pointing in the direction of the steepest slope or grade at that point. The steepness of the slope at that point is given by the magnitude of the gradient vector. The gradient can also be used to measure how a scalar field changes in other directions, rather than just the direction of greatest change, by taking a dot product. Suppose that the steepest slope on a hill is 40%. A road going directly uphill has slope 40%, but a road going around the hill at an angle will have a shallower slope. For example, if the road is at a 60° angle from the uphill direction (when both directions are projected onto the horizontal plane), then the slope along the road will be the dot product between the gradient vector and a unit vector along the road, namely 40% times the cosine of 60°, or 20%.

More generally, if the hill height function f is differentiable, then the gradient of f dotted with a unit vector gives the slope of the hill in the direction of the vector, the directional derivative of f along the unit vector $\vec{l}$.

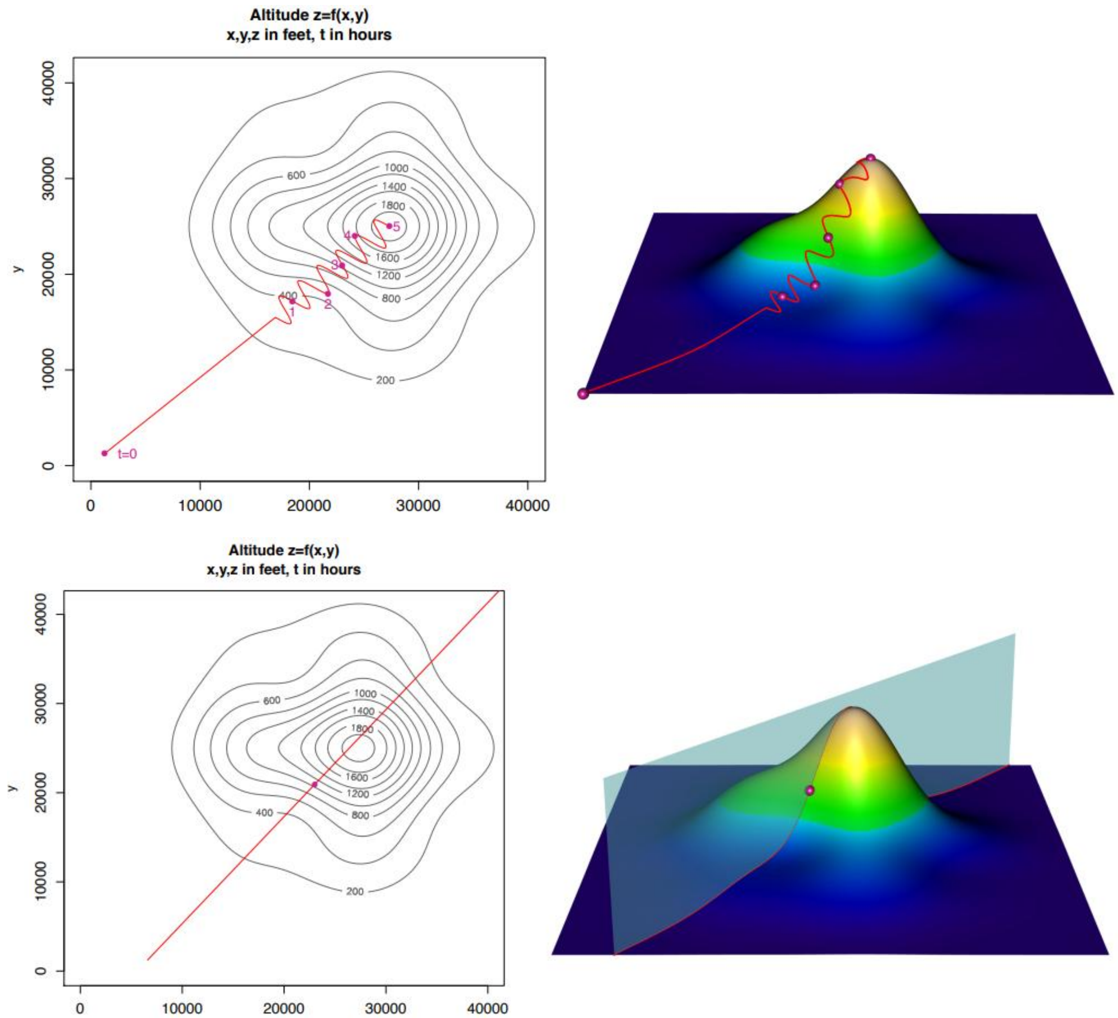


Figure 3.9.

At point P, which direction $\vec{u}$

- The **power hiker** wants the steepest uphill path.
- The **power skier** wants the steepest downhill path.
- The **lazy hiker** wants to avoid any elevation change.

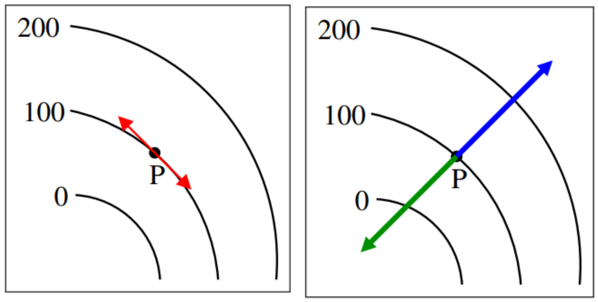


Figure 3.10.

To avoid elevation change, the lazy hiker walks along a level curve. At point P, the direction $\vec{u}$ is tangent to the level curve, giving the two options shown above. There is no elevation change along this path, i.e., $\frac{df}{d\vec{u}} = 0$, so $\text{grad} f \cdot \vec{u} = 0$, so $\text{grad} f \perp \vec{u}$ - the gradient is perpendicular to the level curve.

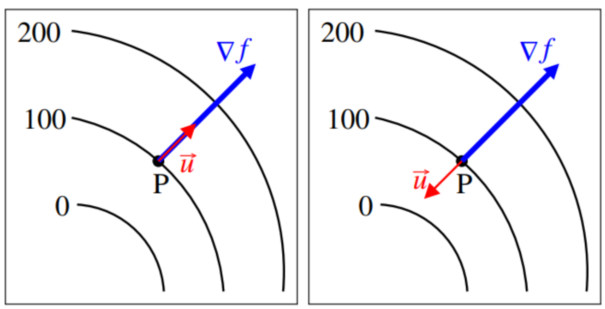


Figure 3.11.

In the case of power hiker, the maximum value of $\frac{df}{d\vec{u}}$ is $+|\text{grad} f|$, thus $\vec{u}$ is a unit vector in the same direction as $\text{grad} f$ and $\vec{u} = \frac{\text{grad} f}{|\text{grad} f|}$. This is the direction of the steepest ascent, or the fastest increase. In the case of power skier, $\vec{u}$ is a unit vector in the opposite direction of $\text{grad} f$ and $\vec{u} = -\frac{\text{grad} f}{|\text{grad} f|}$. This is the direction of the steepest decent, or the fastest decrease.

Tasks:

- Path of steepest ascent: Draw a path starting at a point (yellow), continually adjusting direction to stay perpendicular to the contour in the uphill (increasing) direction.
- Path of steepest descent: Similar but going downhill.

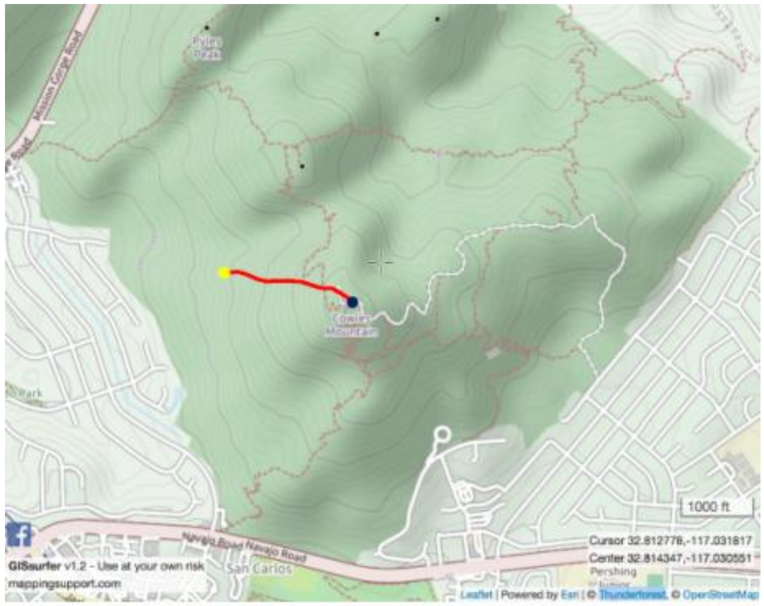


Figure 3.12.

Example 3.1. Suppose that the height of a hill above sea level is given by $f(x, y) = 1000 - 0.01x^2 - 0.02y^2$. If you are at the point $(60{,}100)$ in what direction is the elevation changing fastest? What is the maximum rate of change of the elevation at this point?

Solution. The maximum rate of change of the elevation will then occur in the direction of the gradient vector

$$\text{grad}f(x, y) = (-0.02x, -0.04y), \text{grad}f(60{,}100) = (-1.2, -4).$$

The maximum rate of change of the elevation at this point is,

$$|\text{grad}f(60{,}100)| = \sqrt{(-1.2)^2 + 4^2} = \sqrt{17.44} = 4.176.$$

Let's note that we're at the point $(60{,}100)$ and the direction of greatest rate of change of the elevation at this point is given by the vector $(-1.2, -4)$. Since both of the components are negative, it looks like the direction of maximum rate of change points up the hill towards the centre rather than away from the hill.

6. References

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4. TOPIC: Local, Constrained local and Global extremes

1. Justification for topic choice

The use of constrained extrema of multivariable functions often occurs in optimization problems, where it is necessary to maximize or minimize a given function subject to certain conditions. In practice, constrained extrema of multivariable functions frequently arise in economic, engineering, and mathematical problems, as well as in finding a mathematical model for a given problem in these fields.

2. Historical background

Functions of more variables is a function that has more than one input argument. This concept was introduced in ancient Greece, where mathematicians such as Archimedes and Euclid used multiple variables to describe geometric shapes and motion. The credit for differential calculus can be attributed to the Indian mathematician Bhaskara (1114-1185), who demonstrated an example of what we now know as the differential coefficient and also provided the basic idea of today's Rolle's theorem. The Indian mathematician Madhava, along with other mathematicians of the Kerala school in the 14th century, made many interesting excursions into differential and integral calculus. However, the real breakthrough occurred in the 17th century when Leibniz and Newton worked on what is now considered the discovery of differential and integral calculus. The first written mention of differential calculus can be found in the letters exchanged by Leibniz and L'Hospital in 1695, where Leibniz mentioned the derivation of order 0.5. At that time, he could not predict its significance or the method of computation. However, he noted that one day useful practical consequences would be drawn from this paradox. Other significant scientists also pursued this idea, including giants such as Bernoulli, Euler, Laplace, Fourier, Abel, Riemann, and Cauchy. Nowadays, functions of more variables have an important role in many fields such as mathematics, physics, economics, computer science, engineering, and many other sciences. Multivariable functions are used to model real-world situations and enable computations of complex systems and processes that would be difficult or impossible to solve using function of one variable.

3. Learning outcomes

On completion this module students should be able to

- definition and computation techniques for local extreme of functions of more variables
- how to determine constrained local extrema of function of two variables
- to find the maximum and minimum values of a multivariable function on a given subset M of its domain

- application of knowledge in practice

Prerequisites: Before starting this module students should know

- the principle of differentiating a function of one variable
- functions of more variables
- partial derivatives of functions of more variables

4. Theoretical foundations

- ## Local extreme of functions of two variables

There are two types of local extrema of a function of two variables: local maximum and local minimum. A local maximum of a function $f(x,y)$ occurs at the point (a,b) from the domain if for every point (x,y) from the domain in a certain neighborhood of (a,b), it holds that $f(x,y) \leq f(a,b)$.

A local minimum of a function $f(x,y)$ occurs at the point (a,b) from the domain if for every point (x,y) from the domain in a certain neighborhood of (a,b), it holds that $f(x,y) \geq f(a,b)$.

There are various methods for finding local extrema of a function of two variables, such as the method of partial derivatives, the Hessian matrix, or Lagrange multipliers. These methods allow us to determine points at which the function changes from increasing to decreasing or from decreasing to increasing, and thus determine whether it is a local maximum or minimum.

Let $f(x,y)$ be a function with continuous second partial derivatives in the neighborhood of the point (a,b). The Hessian matrix of the function f is the matrix of second partial derivatives, and the determinant of the Hessian matrix (the Hessian) is calculated as:

$$\begin{vmatrix} \dfrac{\partial f^2(x,y)}{\partial x^2} & \dfrac{\partial f^2(x,y)}{\partial x\,\partial y} \\ \dfrac{\partial f^2(x,y)}{\partial x\,\partial y} & \dfrac{\partial f^2(x,y)}{\partial y^2} \end{vmatrix}$$

Then, it holds that: If the determinant of the Hessian of the function f at the point (a,b) is positive and the second partial derivative with respect to x at the point (a,b) is positive, then f has a local minimum at the point (a,b).

If the determinant of the Hessian of the function $f$at the point (a,b) is positive and the second partial derivative with respect to x at the point (a,b) is negative, then f has a local maximum at the point (a,b).

If the determinant of the Hessian of the function f at the point (a,b) is negative, then $f$does not have a local extremum at the point (a,b). This test allows us to determine the local extrema of a function of two variables, if the function is sufficiently smooth and has continuous second partial derivatives in the neighborhood of the point (a, b). If the determinant of the Hessian is zero, other methods are needed to determine local extrema.

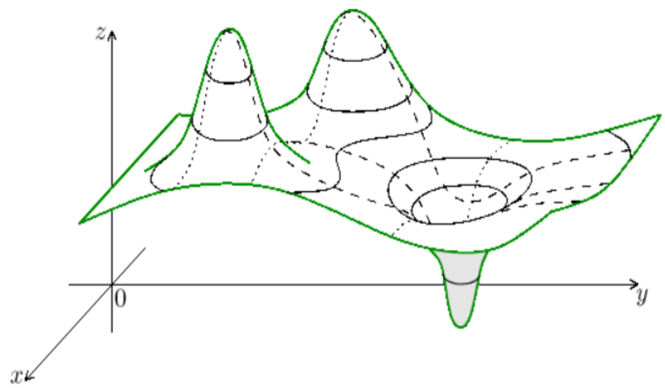

Figure 4.1.

- ## Constrained local extreme of functions of two variables

The problem is to find point A in set M such that the functional value $f(A)$ is the greatest or the least, compared to the values of f at the points of the set M, lying close to the point A. Point A is called a point of the constrained extreme. We define the constrained extreme of functions of two variables:

Let f be function with two variables defined on $D(f) \supset E^2$ and let set

$V = [x, y] \in D(f): g(x, y) = 0 \supset D(f)$ be given. Condition determined by equation

$g(x, y) = 0$ which is satisfied by coordinates of all points from the function f domain of definition $D(f)$ that are in the set $V$is called constraint. Extrema of function f, attained on the set $V \supset D(f)$ determined by constraint are constrained local extrema of function f.

Point $A = [x_0, y_0]$ is called the point of constrained local maximum (minimum) of funct f for the constraint $g(x, y) = 0$, if there exists such neighbourhood $O_\epsilon(A)$ of point A, that for all $X \in O_\epsilon(A)$, whose coordinates satisfy given constraint holds $f(X) \leq f(A)(\big(f(X) \geq f(A)\big)$. In case of strict inequalities we speak about strict constrained local maximum or minimum. Constrained local minimum and maximum of function are together called constrained local extrema of function. How to determine constrained local extrema of function $f(x, y)$?

1. Variable y can be extracted from the constraint $g(x, y) = 0$ and determined as function of variable $x, y = h(x)$ This function can be substituted to the function $f(x, y)$, while a composite function of one variable x defined on the set V can be obtained

   $f\big(x, h(x)\big) = F(x)$. All local extrema of function $F(x)$ on set V are also constrained local extrema of function $f(x, y)$ with two variables on set V.

2. Variable x can be extracted form the constraint $g(x, y) = 0$ and determined as function of variable y $x = h(y)$ This function can be substituted to the function $f(x, y)$, while a composite function of one variable y defined on the set V can be obtained

   $f(h(y), y) = F(y)$ All local extrema of function $F(y)$ on set V are also constrained local extrema of function $f(x, y)$ with two variables on set V.
3. In case, none from variables x or y can be extracted from the constraint $g(x, y) = 0$ and expreessed in terms of the other, the method of Lagrange multipliers can be used. We define an auxhiliary function called Lagrange function $L(x, y) = f(x, y) + \lambda g(x, y)$, where λ is an arbitrary constant called Lagrange multiplier. Function $L(x, y)$ is defined on set $D(f)$, and moreover, in all points of the set V holds $L(x, y) = f(x, y)$ as $g(x, y) = 0$ in the points of set V.

If any point $A = [x_0, y_0] \in V$ is the point of local extremum of function $L = f + \lambda g$, then point A is the point of constrained local extremum of function f for the constraint$g(x, y) = 0$. Geometric interpretation Constrained local extrema of function f are $z$coordinates of extremely located points on curve, which is intersection of graph $G(f)$ of function f with the cylindrical surface determined by curve defined in the plane xy by constraint, while lines on this surface are in direction of coordinate axis z.

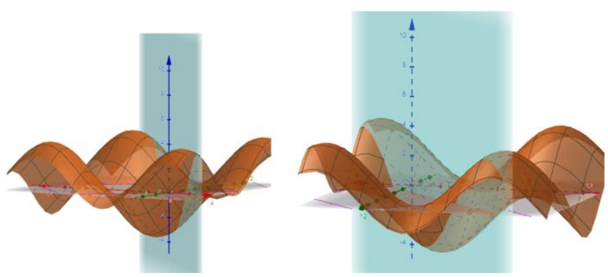

Figure 4.2.

- ## Global extrema of a multivariable function

To find the maximum and minimum values of a multivariable function on a given subset M of its domain. Determine and select the maximum and minimum values of the function on M by: i) all local extrema of the function of two variables on the subset M ii) all the constrained local extrema on the boundary of M iii) finding the values on the boundary of M where the function attains its maximum and minimum values.

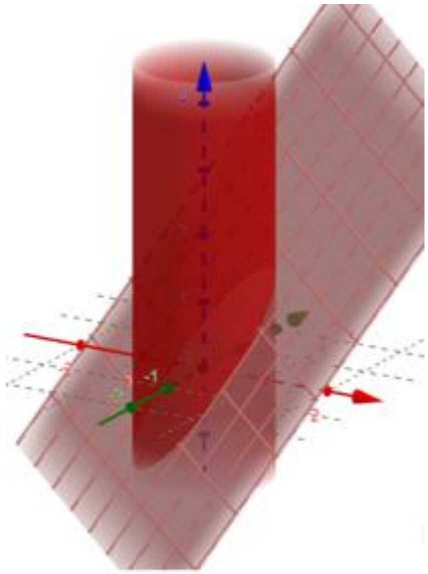

Figure 4.3.

5. Applications to everyday life

- In trading, it can be useful to pay special attention to extreme price levels - such as the highest and lowest prices in the market - in order to respond to buying or selling opportunities. In finance, extreme values can also relate to risk. For example, risk capital may be used to finance new and innovative projects, but it can also be associated with high losses and failure.
- In medical practice, extreme cases are often crucial for determining a patient's diagnosis and treatment. Doctors must be able to recognize and treat patients with the most severe symptoms or conditions.
- In industrial production, extreme values can be part of quality control and monitoring processes. For example, industrial equipment may be designed to operate at extreme temperatures or under pressure to ensure high production quality.
- In the field of computer security, extreme cases can relate to the most dangerous threats and attacks on systems. Security teams must be able to recognize and address these most serious threats.
- In marketing, extreme cases can relate to the most successful campaigns and strategies that led to increased sales or brand awareness. Analyzing these cases can be useful in designing new campaigns.
- In insurance and financial planning, extreme cases related to accidents or loss of property can be key to assessing risk and determining an insurance plan.
- In meteorology, extreme cases such as hurricanes, tornadoes, or floods are important for weather forecasting and issuing warnings. Meteorology teams must be able to identify and address extreme cases to minimize damage.
- In science and research, extreme cases can be crucial for discovering new knowledge and developing scientific theories. For example, extreme conditions such as very low or high temperatures can be used to test material properties and discover new scientific breakthroughs.
- In sports and fitness, extreme cases relate to the best performances and records. Training plans can be designed to maximize athletes' performance and lead to extreme results, such as Olympic medals or world records.

6. References

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5. TOPIC: Cross-sections of Solids

1. Justification for topic choice

The study of cross-sections of a polyhedron (sections defined by the intersection of a plane with the polyhedron) is a fundamental concept in geometry. This technique is widely used to explore the solids' geometric properties and to analyze the shape and structure of the resulting planar sections [Anwar, N., and Najam, F. A. (2017)]. It enhances students' understanding of three-dimensional geometry by establishing a crucial connection between 2D and 3D geometry, promoting a deeper interpretation of spatial relationships. Cross-sections are highly relevant across numerous STEM fields. As noted by P. Lewis, [P.Lewis,2016], the cross-section perspective offers two distinct architectural representations simultaneously: the section reveals hidden details such as wall thickness and vertical organization, while the perspective allows the viewer to interpret the effects of the section from a single vantage point. Cross-sections are also extensively used in engineering, they are essential for understanding the structure and behavior of various section shapes. Through both parametric and general methods, engineers calculate geometric properties and analyze complex and composite sections, which are critical for improving the performance of engineering systems [Smith, 2019]. Additionally, cross-sections are used in computer graphics [Schumaker, 1990], especially for creating realistic and detailed 3D models. By slicing a 3D object, we obtain essential data to calculate specific measurements needed for a given purpose.

2. Historical background

Ancient Egyptians and Mesopotamians were among the first to use geometry practically, particularly for land measurement and architecture. The Egyptians, known for their skills in building monumental structures such as the pyramids, would likely have had an implicit understanding of cross-sections. Although their knowledge was essentially acquired as applied science, it laid the basis for the foundations of the geometric concepts that would later expand through Greek and Hellenistic thinkers. The study of geometry became more formalized, with mathematicians like Euclid, who developed a systematic approach to geometry, expressed in his valuable work, Elements. Euclid's studies included slicing shapes to explore their properties, laying a foundation for later work on cross-sections. Greek mathematicians saw geometric shapes as objects of mathematical analysis and symbols of philosophical thought, and the ability to "slice" through shapes to examine their interiors was a radical idea that mirrored their broader quest for understanding the unseen structures of reality. In the Renaissance period, as art, science, and architecture flourished, the study of cross-sections evolved even further. Artists and scientists like Leonardo da Vinci and architects of the time used cross-sections to plan buildings, study anatomy, and create more accurate drawings. Leonardo's anatomical studies, for instance, involved dissecting both bodies and objects, providing him with insights into how they worked from the inside out. Architects also began to use cross-sections to visualize the inner structure of complex buildings, helping them create designs that were not just beautiful but structurally sound. Kepler's work also included the study and analysis of cross-sections. While Kepler is widely known for his contributions to planetary motion, his lesser-known work on the measurement of volumes, particularly in irregular solids, played a crucial role in advancing mathematical methods that anticipated integral calculus. In his work, *Nova Stereometria Doliorum Vinariorum*, Kepler tackled the problem of determining the volume of wine barrels, which have a complex, curved shape. Kepler realized that traditional geometric formulas for volume did not apply well to these irregular shapes. To address this, he began "slicing" the barrels conceptually, dividing them into an infinite series of thin cross-sections. By summing the areas of these cross-sections, he could approximate the volume of the entire barrel. This method of calculating volume by summing slices resembles the concept of integration, which would later be formalized by Isaac Newton and Gottfried Leibniz in the development of calculus. Today, cross-sections are an essential part of modern life, going far beyond mathematics and physics. The integration of technology allows, for example, architects and engineers to create detailed digital models of cross-sections, allowing greater robustness to the structures they are working with. In short, cross-sections have evolved from a resource for understanding shapes to an important modern resource that connects our past to our present, driving innovations in fields as diverse as health, technology, and design.

3. Learning outcomes

Upon completing this module, students should be able to:

- Develop the ability to visualize and interpret the shapes formed by the intersections of a plane with a polyhedron;
- Perform rotations and mental manipulations of solids, predicting the shape of the cross-section produced in a polyhedron, without using physical or digital artifacts;
- Recognize the relationship between three-dimensional figures and their two-dimensional sections;
- Recognize how the symmetry of a polyhedron affects the shape of the resulting cross-section;
- Think and reason, geometrically, to model solids resulting from sections produced by different cuttings in a polyhedron.
- Describe cross-sections accurately using geometric terminology (e.g., polygons, parallel, perpendicular);
- Communicate findings and predictions about cross-sections through verbal explanations, drawings, or models.
- Solve problems involving the metric and geometric properties of the cross-sections.
- Explore distinct methods to achieve congruent cross-sections.
- Analyze, critically, the effect of the plane's orientation on the resulting cross-section

- Explore cases with unexpected outcomes.

Prerequisites: The prerequisites to be observed in this module are (1) familiarity with basic 2D geometric shapes (such as polygons and circles) and 3D solids (such as cubes, cylinders, cones, and spheres); (2) understanding the relationship between faces, edges, and vertices, essential for visualizing and describing 3D shapes); (3) knowledge of geometric results involving plane intersections. For a more advanced study of cross-sections, (4) a basic understanding of the 3D coordinate system (x, y, z) can be useful to analyze where and how a plane intersects a solid. This is particularly useful when moving to irregularly shaped cross sections or understanding the mathematics behind slicing. A (6) preliminary understanding of transformations, such as rotations and reflections in 2D and 3D space, can also help. These concepts are foundational to more advanced geometry topics, including symmetry and, later, the classification of cross-sections in different situations, where (7) spherical coordinates will be of great importance.

4. Theoretical foundations

To understand cross-sections of polyhedra, it's crucial to consider propositions involving relationships between lines and planes. Key theorems are listed below.

Theorem 5.1. *The intersection of any two planes in space is empty or a straight line.*

Theorem 5.2. *A line l is parallel to a plane α if and only if there exists a line $m \subset \alpha$ such that l is parallel to m.*

As a straightforward result, we have:

- *If a line ℓ is parallel to the intersection of two planes, then ℓ is also parallel to each of those planes*.

Theorem 5.3. *If a straight line is parallel to a plane, every plane that contains the straight line and intersects the original plane does so along a straight line parallel to the original straight line.*

This proposition helps predict the orientation of cross-sections. For example, if a cutting plane intersects a polyhedron and is parallel to one of its edges, the resulting cross-section will have lines parallel to that edge, see Figure 5.1.

Theorem 5.4. *Two planes are parallel if and only if one of them contains a pair of concurrent lines that are parallel to the other plane.*

Theorem 5.5. *If a plane intersects two parallel planes, the intersections are two parallel lines.*

Let’s explore some of the cross-sections produced by intersecting a plane with a tetrahedron, a cube, or a dodecahedron. These convex polyhedra are part of the family of the five Platonic solids, solids whose faces are congruent regular polygons. This family is made of tetrahedra, cubes, octahedra, dodecahedra, and icosahedra, see Fig 2. Known since antiquity, these solids were studied in depth by Plato, who associated each of them with the fundamental elements of nature: earth (cube), air (octahedron), fire (tetrahedron), water (icosahedron), and the cosmos (dodecahedron).

We may associate to each platonic solid a **Schläfli symbol** given by two entries,p, q, where p is the number of sides on each face, and q is the number of faces meeting at each vertex. For instance, the icosahedron has the Schläfli symbol, 3,5, as each face is an equilateral triangle (3), and five equilateral triangles meet at each vertex. For the dodecahedron, the Schläfli symbol is 5,3, as each face is a regular pentagon (5), with 3 of them meeting at each vertex. We may also associate to each of them its **vertex configuration**, which is, the sequence of faces around a vertex. Thus, the **vertex configuration** of the icosahedron is 3.3.3.3.3 and the **vertex configuration** of the dodecahedron is 5.5.5. The combinatorial description of the Platonic solids is shown in the following table.

Polyhedron		Vertices	Edges	Faces	Schläfli symbol	Vertex configuration
Regular tetrahedron		4	6	4	{3, 3}	3.3.3
cube		8	12	6	{4, 3}	4.4.4
Regular octahedron		6	12	8	{3, 4}	3.3.3.3
dodecahedron		20	30	12	{5, 3}	5.5.5
icosahedron		12	30	20	{3, 5}	3.3.3.3.3

Figure

Figure 5.1. https://en.wikipedia.org/wiki/Platonic_solid

- ## Cross-sections in a tetrahedron

Let us first state some geometric properties of the tetrahedron.

Theorem 5.6 *Let $\mathcal{T}$ be a tetrahedron with edge length a, Then:*

- *the height h of $\mathcal{T}$ is given by, $h = \frac{a\sqrt{6}}{3}$;*

- *the total area A_T of $\mathcal{T}$ is given by, $A_T = a^2\sqrt{3}$;*
- *the volume V of $\mathcal{T}$ is given by, $V = a^3\frac{\sqrt{2}}{12}$.*

Theorem 5.7. *A (regular) tetrahedron has 24 distinct symmetries, consisting of 12 rotational symmetries, 6 (pure) reflections, and 6 rotary reflections.*

Any symmetry of the tetrahedron maps vertices onto vertices and thus can be represented by a permutation of its vertices. From now on, let us assume that $\mathcal{T}$ has vertices A, B, C, and D.

- **The 12 rotational symmetries** which preserve orientation, include:
  a) **the identity**;
  b) **8 rotations** of 120° and 240° around axes passing through each of the four vertices and the centers of the opposite faces; and
  c) **3 rotations of 180°** around axes passing through the midpoints of opposite edges.

(1) The **identity symmetry** is represented by the permutation: $\begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}$.

Let us now characterize the rotations of 120° and 240 around the axis passing through the vertice D and the center $C1$ of the opposite face , $[A, B, C]$,see Figure 5.2.

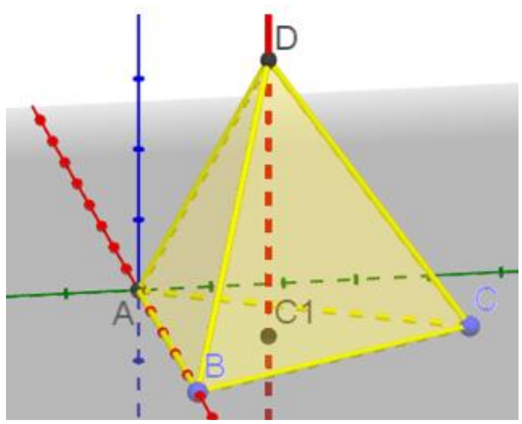

Figure 5.2.

Figure 5.2. Rotations about an axis joining a vertex to the center of the opposite face.

The corresponding representations, in terms of permutations, are as follows:

$$(2)\ \begin{pmatrix} A & B & C & D \\ B & C & A & D \end{pmatrix} \qquad (3)\ \begin{pmatrix} A & B & C & D \\ C & A & B & D \end{pmatrix}.$$

There are 4 vertices, so we have four axes of rotations, see Figure 5.3.

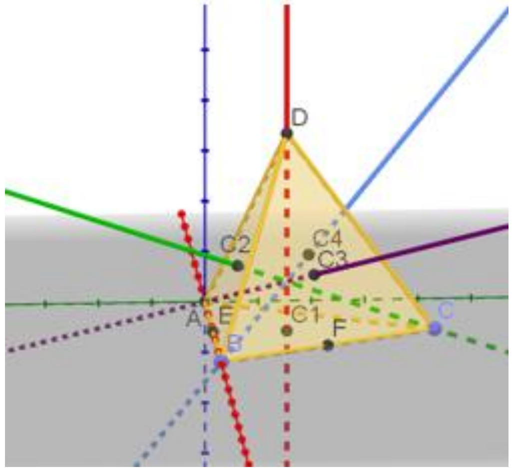

Figure 5.3. The 4 axes of rotation joining a vertex to the center of the opposite face.

The permutations corresponding to the rotations about the green, blue, and violet axes, are:

$$(4)\ \begin{pmatrix} A & B & C & D \\ B & D & C & A \end{pmatrix} \qquad (5)\ \begin{pmatrix} A & B & C & D \\ D & A & C & B \end{pmatrix};$$

$$(6)\ \begin{pmatrix} A & B & C & D \\ C & B & D & A \end{pmatrix} \qquad (7)\ \begin{pmatrix} A & B & C & D \\ D & B & A & C \end{pmatrix};$$

$$(8)\ \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix} \qquad (9)\ \begin{pmatrix} A & B & C & D \\ A & D & B & C \end{pmatrix};$$

We have three more half-turn rotations, through the axes passing through the midpoints of two opposite edges, see Figure 5.4.

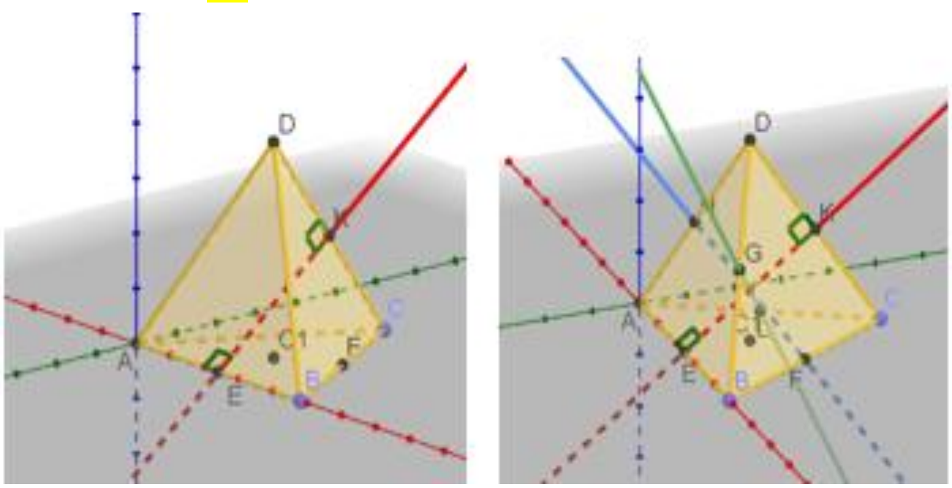

Figure 5.4. Rotations about axes joining midpoints of opposite faces.

The permutations corresponding to the half-turns about the red, blue, and green axes are, respectively:

$$(10)\ \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix} \qquad (11)\ \begin{pmatrix} A & B & C & D \\ D & C & B & C \end{pmatrix} \qquad (12)\ \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}.$$

The 12 reflectional symmetries, which do not preserve orientation, consist of:

a) **6 (pure) reflections,** each on a plane passing through an edge and the midpoint of the opposite edge; and

b) **6 rotatory reflections**, each formed by combining a rotation with a pure reflection.

Let us characterize **the 6 pure reflections**. Consider the reflection in the plane containing the edge $[BC]$ and the midpoint of $[AD]$, see Figure 5.5. It fixes B and C and permutes the other two vertices. Therefore, it can be represented by the permutation,

$$(1)\ \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}.$$

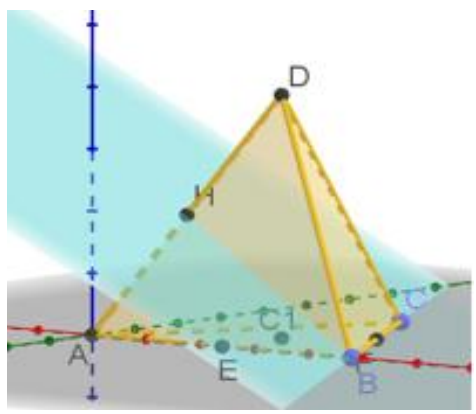

Figure 5.5. Reflection in a plane containing an edge and passing through the midpoint of the opposite edge.

Since the tetrahedron has 6 edges, there are 5 additional reflection planes. The following permutations represent these additional reflectional symmetries:

$$(2)\ \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}; \quad (3)\ \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}; \quad (4)\ \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix};$$

$$(5)\ \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}; \quad (6)\ \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}.$$

The 6 rotatory reflections symmetries of the tetrahedron are the ones represented by the permutations given below. For each of these, we will present a combination involving a pure reflection and a rotation.

$$(1)\ \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix} \circ \begin{pmatrix} A & B & C & D \\ B & C & A & D \end{pmatrix}.$$

$$(2)\ \begin{pmatrix} A & B & C & D \\ C & D & B & A \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A & B & D & C \end{pmatrix} \circ \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}.$$

$$(3)\ \begin{pmatrix} A & B & C & D \\ D & C & A & B \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A & B & D & C \end{pmatrix} \circ \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}.$$

$$(4)\ \begin{pmatrix} A & B & C & D \\ B & D & A & C \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix} \circ \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}.$$

$$(5)\ \begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix} \circ \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}.$$

$$(6)\ \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix} \circ \begin{pmatrix} A & B & C & D \\ D & A & C & B \end{pmatrix}.$$

Since the group of permutations of four elements has exactly 24 elements, we can conclude that the symmetry group of the tetrahedron is fully determined.

Let's now explore the cross-sections that can be obtained, intersecting a plane with a tetrahedron. To do so, we will utilize three different approaches, each tailored to the prerequisites that readers of this module may have. We will begin with the **naive approach**, which requires only minimal prerequisites. Following this, we'll introduce the **intermediate approach**, which assumes a basic understanding of isometric transformations in space. Finally, we will present the **advanced approach**, where a solid understanding of the symmetry group is essential, along with dexterity in spherical coordinates.

- ## Cross-sections in a tetrahedron - Naive approach

Let us consider the tetrahedron $\mathcal{T}$ with vertices A, B, C, and D. Since the faces of $\mathcal{T}$ lie within a plane, by Theorem 5.1, if a plane intersects a face of $\mathcal{T}$, it must do so either at a single vertex or along a line segment that connects two edges of that face. Consequently, since three non-collinear points define a plane, to determine the cutting plane, it suffices to select defining points that lie on the (closed) edges of the tetrahedron.

Let P, Q and R be three non-collinear points in
$\mathcal{E} = [A\,B] \cup [B\,C] \cup [A\,C] \cup [A\,D] \cup [B\,D] \cup [C\,D]$, and π the plane defined by them.

Assuming that P and Q are points belonging to the same edge e of the tetrahedron, being possibly the endpoints of that edge, then R necessarily lies on the opposite edge e', and the cross-section produced in $\mathcal{T}$ by the plane π is a triangle, see Figure 5.6.

This triangle is always isosceles, regardless of the specific positions of P and Q on edge e and R on edge e'.

Why does this happen?

Observe that the triangles $[ADR]$ and $[BDR]$ are congruent by the Side-Angle-Side (SAS) congruence criterion.

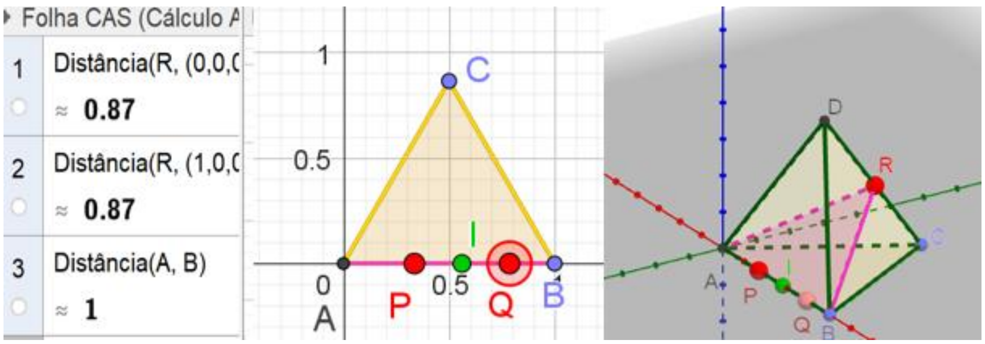

Figure 5.6. The cross-section produced by the plane passing through P, Q and R.
(https://geogebratools.pages.dev/tetraedro-naive - Ana Breda)

Furthermore, the triangular cross-section cannot be equilateral because the angle at A is less than $\frac{\pi}{6}$ rad.

- Assume, now, that P, Q, and R are three points such that no two of them lie on the same edge, and one of them, for example P, is a vertex of the tetrahedron. Then the cross-section is an isosceles or a scalene triangle, as shown, respectively, in Figure 5.7.a and 5.7.b.

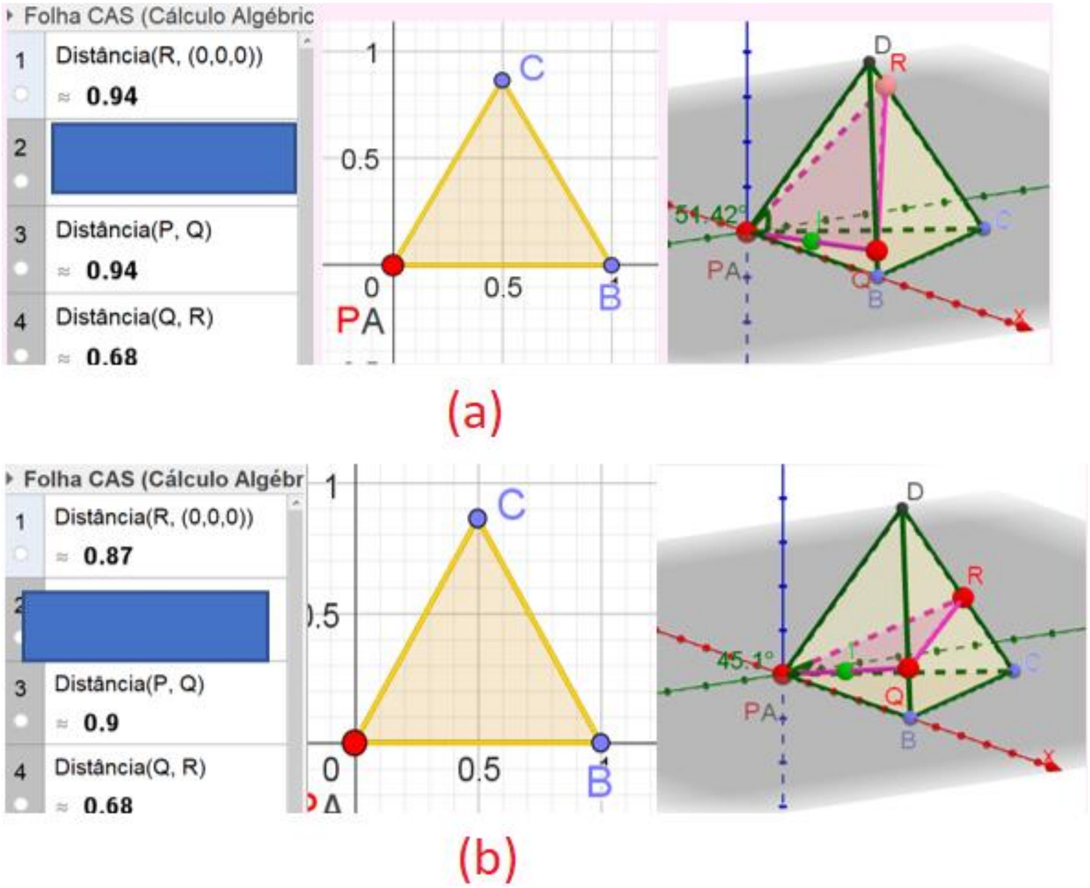

Figure 5.7. The cross-section produced by the plane passing through P, Q and R.
(https://geogebratools.pages.dev/tetraedro-naive - Ana Breda)

Is it possible to obtain an equilateral triangular cross-section under the given assumption? Use the applet available at (https://geogebratools.pages.dev/tetraedro-naive to examine the behavior of angle beta and the side lengths of the triangle in the cross-section. Based on your observations, construct a mathematical justification that leads to the conclusion that, under the given assumptions, an equilateral cross-section cannot be achieved.

- Assume now that P, Q and R do not lie on the same edge of the tetrahedron; none of these points coincide with any vertex of the tetrahedron, and all of them lie on edges that emerge from the same vertex.

To analyze the type of polygon formed in the cross-section of the tetrahedron by the plane defined by points P, Q and R use the GegoGebra applet (https://geogebratools. pages.dev/tetraedro-naive) position the points P, Q and R as required and describe your observations, see Figure 5.8. Explain the results you have observed using mathematical reasoning.

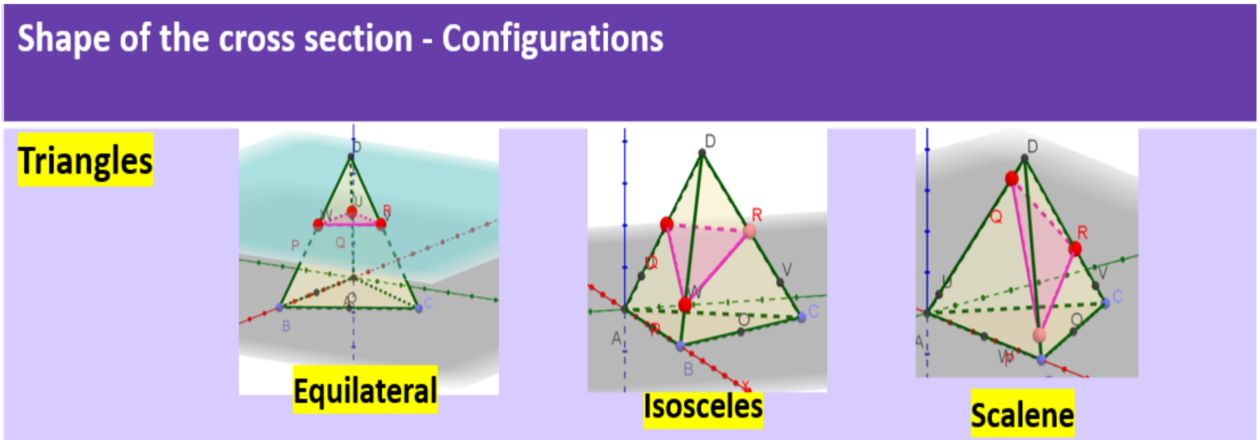

Figure 5.8. The cross-section produced by the plane passing through P, Q and R.
(https://geogebratools.pages.dev/tetraedro-naive - Ana Breda)

- Finally, assume that P, Q, and R are such that any two of them do not lie on the same edge, none of them is a vertex of the tetrahedron, and not all of them belong to edges emerging from the same vertex.

Use the same application and carefully observe the formation of the cross-sections, see Figure 5.9.

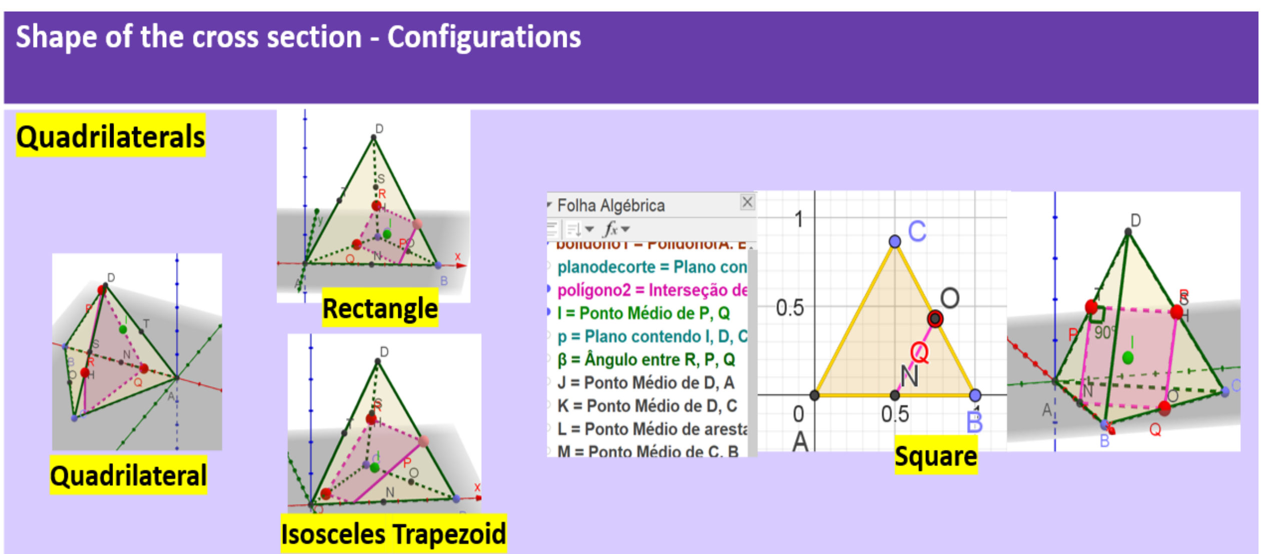

Figure 5.9. The cross-section produced by the plane passing through P, Q and R.
(https://geogebratools.pages.dev/tetraedro-naive - Ana Breda)

Based on your simulations and observations, could you present mathematical arguments supporting these observations?

- ## Cross-sections in a tetrahedron - Intermediate approach

The study of cross-sections by planes in a tetrahedron can be significantly systematized using the concept of symmetry. In Theorem 5.7, we described the symmetry group of the tetrahedron. In this approach, we only need to consider one of its subgroups, for instance, the subgroup generated by a 120-degree rotation around the axis passing through a vertex and the center of the opposite face, see Figure 5.10.

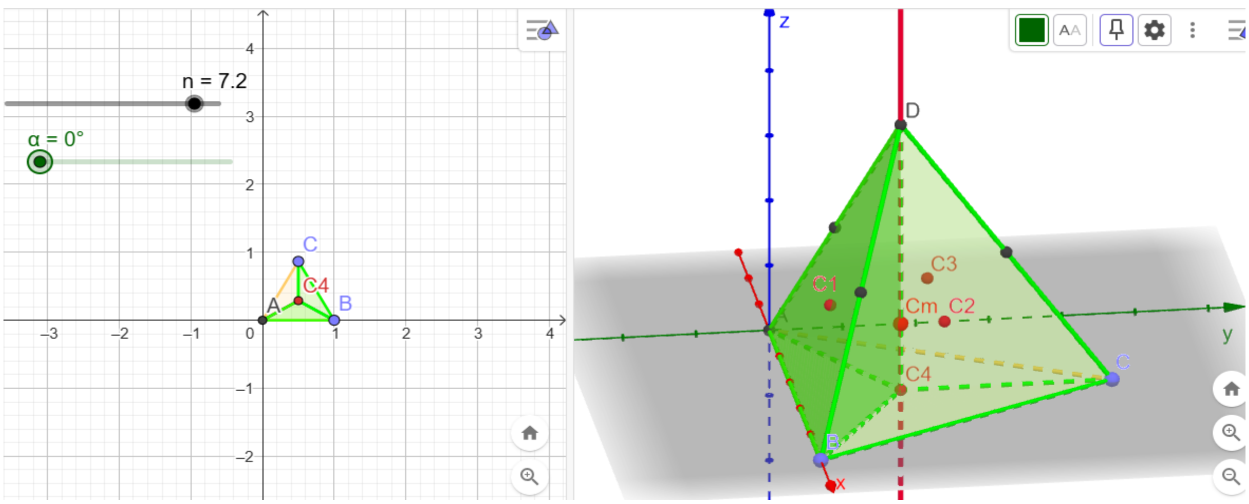

Figure 5.10: The cross-section produced by the plane passing through P, Q and R.
(https://geogebratools.pages.dev/tetraedro-intermedia - Ana Breda)

To ensure that no plane is forgotten, it is sufficient to consider the unit vectors in the spherical limit of the pyramid with vertices A, B, C_4, D. For each of these vectors, consider a plane perpendicular to it that passes through a point that belongs to the edges of the pyramid, and finally, consider the family of planes that are parallel to it. **Spherical coordinates** were used to describe the spherical region.

Use part II of the GeoGebra applet to observe, systematically, the formation of cross-sections, see Figure 5.11.

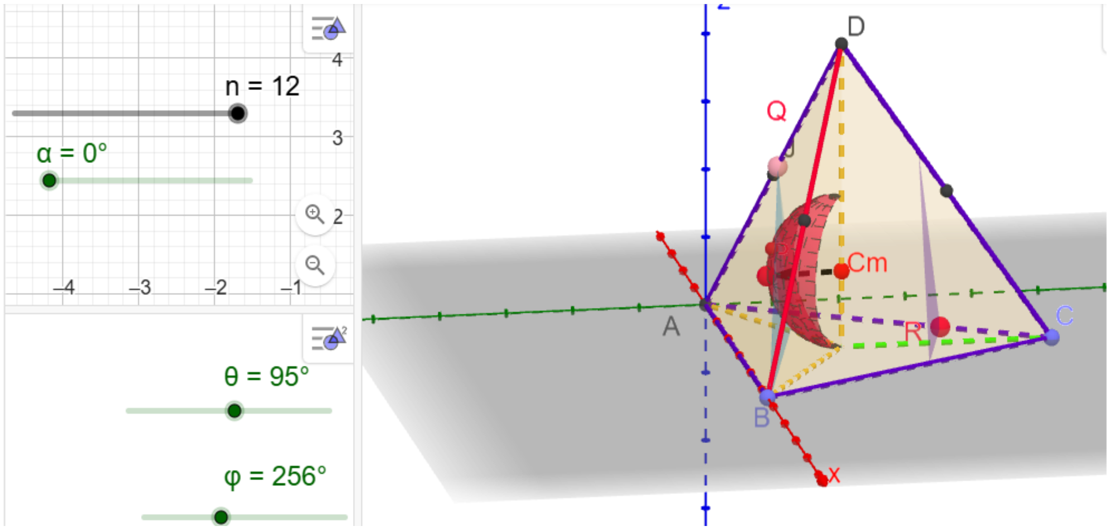

Figure 5.11. The cross-section in a tetrahedron - Intermediate approach.
(https://geogebratools.pages.dev/tetraedro-intermedia - Ana Breda)

- ### Cross-sections in a tetrahedron - Advanced approach

Continuing with this line of thought, we can further filter the study of cross sections, limiting our analysis to a representative region of the tetrahedron, the fundamental region, significantly reducing the number of simulation cases to be carried out. This approach takes advantage of the fact that all possible configurations can be obtained by working only in the fundamental region. It allows obtaining a more efficient classification of potential cross-sections since each configuration within the fundamental region represents a class of equivalent cross-sections throughout the tetrahedron.

A fundamental region may be described by the pyramid with vertices at the centroid of the tetrahedron, the incenter, and the midpoints of one of its faces, see Figure 5.12. By acceding to https://geogebratools.pages.dev/regiao-fundamental-tetraedro we can explore how the symmetry group acts on the fundamental region.

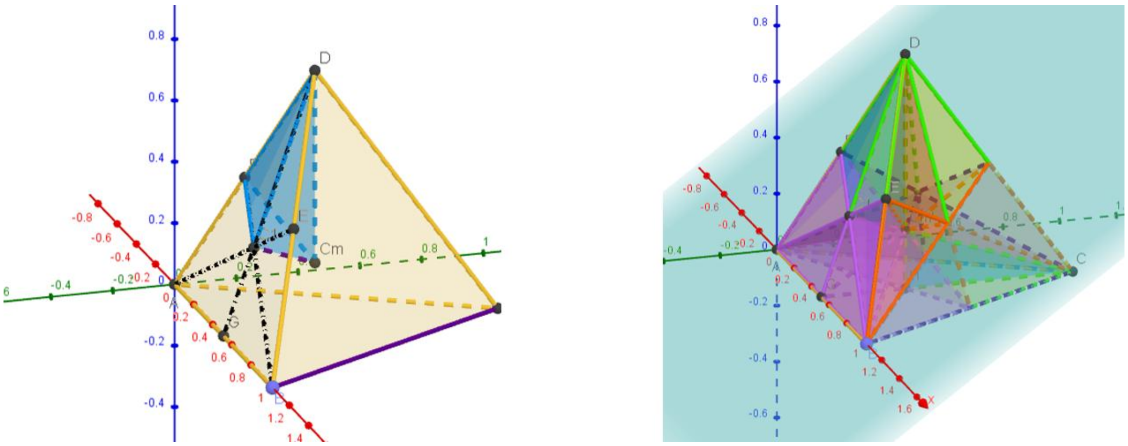

Figure 5.12: A fundamental region of the tetrahedron.
(https://geogebratools.pages.dev/regiao-fundamental-tetraedro - Ana Breda)

We can now apply the procedures carried out in the intermediate approach, repositioned at the level of the fundamental region. This approach allows us to focus on a smaller, yet fully representative region of the tetrahedron, reducing computational complexity and maintaining total analytical rigor, see Figure 5.13. By focusing on the fundamental region, we speed up the study of cross sections, since each configuration obtained from information within this region effectively represents an entire set of equivalent configurations in the tetrahedron.

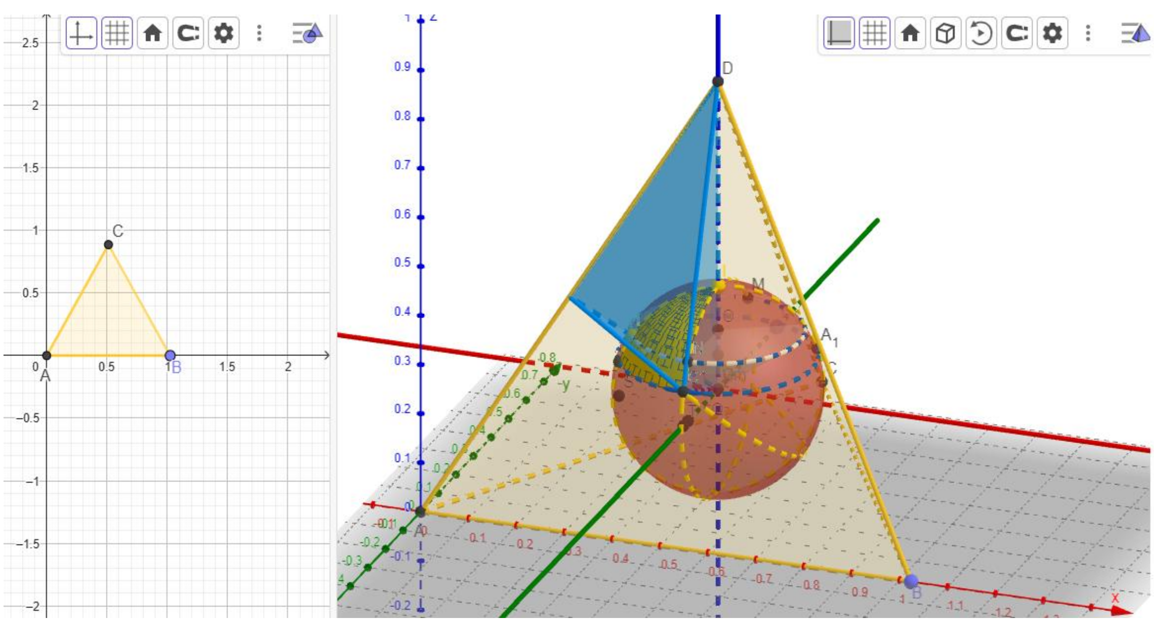

Figure 5.13. The cross-section in a tetrahedron - Advanced approach.
(https://geogebratools.pages.dev/tetraedro-avancado - Ana Breda)

By using the concept of symmetry we not only simplify the mathematical treatment but also improve our understanding of the two-dimensional geometric structures (cross-sections) that are being generated.

5. Applications to everyday life

Understanding the cross-sections of solids isn't just something engineers or scientists do, it's a concept that supports many aspects of our everyday lives. In architecture and construction, cross-sectional views help designers and builders plan strong, stable structures. When architects design cross-sections of beams or columns, they consider not only the aesthetic qualities but also the strength and durability of the structure, ensuring that buildings, bridges, and houses will last and withstand the wear and tear they are subjected to, day after day. In healthcare, cross-sections play a vital role. Techniques like MRI (*Magnetic Resonance Imaging*) and CT (*Computed Tomography*) scans provide a detailed look inside the human body, giving doctors a "slice-by-slice" view of our organs, tissues, and bones. This helps them spot health issues early and make accurate diagnoses, often without the need for invasive surgery. In manufacturing, examining a product's cross-section can reveal hidden flaws or weaknesses. This attention to detail ultimately means safer, higher-quality products for consumers. Geologists use cross-sections of the Earth's layers to understand what lies beneath the surface. When mining companies look for minerals or oil, they rely on these cross-sectional maps to know where to drill or dig. This knowledge helps us access important resources while minimizing environmental disruption. Cross-sections are also powerful tools in the classroom. They help students visualize the shapes and properties of 3D objects in a way that's easier to understand. Many everyday items, such as pipes, cables, cans, and bottles, are designed with specific cross-sectional shapes for practical reasons. For example, a round tube enables water to flow smoothly and efficiently through it, while a cylindrical tube can maximize storage capacity while remaining resistant.

6. References

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6. TOPIC: Vectors and their properties

1. Justification for topic choice

Vectors play an important role in various fields, including physics, engineering, computer graphics, and machine learning, where they are used for calculations and modeling of physical phenomena and mathematical operations. In engineering, frequent reference is made to physical quantities, such as force, speed and time. For example, we talk of the speed of a car, and the force in a compressed spring. It is useful to separate these physical quantities into two types. Quantities of the first type are known as scalars. These can be fully described by a single number known as the magnitude. Quantities of the second type are those which require the specification of a direction, in addition to a magnitude, before they are completely described. These are known as vectors. A vector is a quantity that has a magnitude and a direction. Quantities that are vectors must be manipulated according to certain rules. Special methods have been developed for handling vectors in calculations, giving rise to subjects such as vector algebra, vector geometry and vector calculus. Moreover, vectors help us determine position and change in position of points. In various areas of physics and mathematics, vectors are used to understand the behavior of directional quantities in two and three dimensional spaces.

2. Historical background

Vectors were born in the first two decades of the 19th century with the geometric representations of complex numbers. Mathematicians and scientists worked with and applied these new numbers in various ways. In 1837, William Rowan Hamilton $(1805 - 1865)$ showed that the complex numbers could be considered abstractly as ordered pairs (a, b) of real numbers. This idea was a part of the campaign of many mathematicians to search for a way to extend the two-dimensional "numbers" to three dimensions; but no one was able to accomplish this, while preserving the basic algebraic properties of real and complex numbers. The development of the algebra of vectors and of vector analysis as we know it today was first revealed in sets of remarkable notes made by J. Willard Gibbs $(1839 - 1903)$.

Historically, vectors were introduced in geometry and physics (typically in mechanics) for quantities that have both a magnitude and a direction, such as displacements, forces and velocity. Such quantities are represented by geometric vectors in the same way as distances, masses and time are represented by real numbers.

3. Learning outcomes

On completion this module students should be able to

- categorize a number of common physical quantities as scalar or vector

- represent vectors by directed line segments
- combine, or add, vectors using the triangle law
- resolve a vector into two perpendicular components

Prerequisites: Before starting this module students should

- be familiar with all the basic rules of algebra

4. Theoretical foundations

Definition 6.1. A vector is an object that has both a magnitude and a direction.

Figure 6.1.

Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. The direction of the vector is from its tail to its head.

In a graphical sense vectors are represented by directed line segments. The length of the line segment is the magnitude of the vector and the direction of the line segment is the direction of the vector. However, because vectors don't impart any information about where the quantity is applied any directed line segment with the same length and direction will represent the same vector.

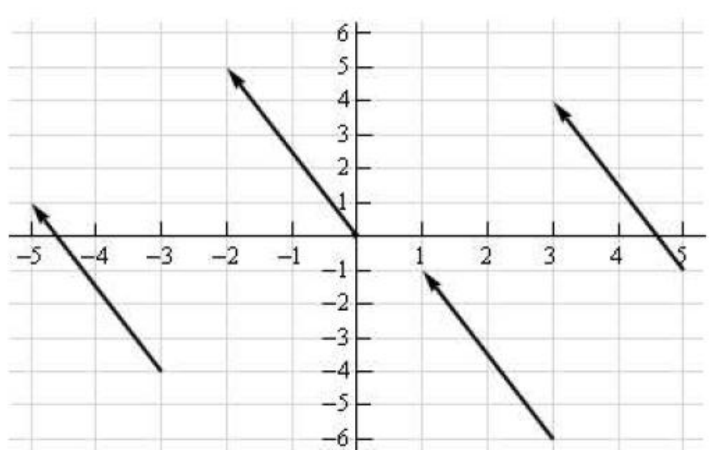

Figure 6.2.

Each of the directed line segments in the figure 6. 2 represents the same vector. In each case the vector starts at a specific point then moves 2 units to the left and 5 units up. The notation that we'll use for this vector is,

$$\vec{v} = (-2{,}5)$$

and each of the directed line segments in this figure are called representations of the vector.

A representation of the vector $\vec{v} = (a_1, a_2)$ in two dimensional space is any directed line segment, $\overrightarrow{AB}$, from the point $A = (x, y)$ to the point $B = (x + a_1, y + a_2)$. Likewise a representation of the vector $\vec{v} = (a_1, a_2, a_3)$ in three dimensional space is any directed line segment, AB , from the point $\overrightarrow{AB}$, from the point $A = (x, y, z)$ to the point $B = (x + a_1, y + a_2, z + a_3)$.

The representation of the vector $\vec{v} = (a_1, a_2, a_3)$ that starts at the point $A = (0{,}0{,}0)$ and ends at the point $B = (a_1, a_2, a_3)$ is called the **position vector** of the point (a_1, a_2, a_3). So, when we talk about position vectors we are specifying the initial and final point of the vector.

We need to discuss how to generate a vector given the initial and final points of the representation. Given the two points $A = (a_1, a_2, a_3)$ a and $B = (b_1, b_2, b_3)$ the vector with the representation $\overrightarrow{AB}$ is,

$$\vec{w} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$$

We have to be very careful with direction here. The vector above is the vector that starts at A and ends at B. The vector that starts at B and ends at A, i.e. with representation $\overrightarrow{BA}$ is,

$$\vec{w} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

These two vectors are different and so we do need to always pay attention to what point is the starting point and what point is the ending point. When determining the vector between two points we always subtract the initial point from the terminal point.

Definition 6.2. The magnitude, or length, of the vector $\vec{v} = (a_1, a_2, a_3)$ is given by,
$\| \vec{v} \| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$.

Definition 6.3. Any vector with magnitude of 1, i.e. $\| \vec{u} \| = 1$, is called a unit vector.

Definition 6.4. The vector $\vec{w} = (0{,}0{,}0)$ with magnitude of 0, i.e. $\| \vec{w} \| = 0$, is called a zero vector.

Remark 6.1. Zero vectors are often denoted by $\vec{0}$. Be careful to distinguish 0 (the number) from $\vec{0}$ (the vector). The number 0 denotes the origin in space, while the vector $\vec{0}$ denotes a vector that has no magnitude or direction.

Definition 6.5. The vectors $\vec{\imath} = (1,0,0), \vec{\jmath} = (0,1,0), \vec{k} = (0,0,1)$, are called a standard basis vectors.

Definition 6.6. A vector having the same magnitude as that of a given vector $\vec{v}$ and the direction opposite to that of $\vec{v}$ is called the negative of $\vec{v}$ and it is denoted by $\overrightarrow{-v}$.

Definition 6.7. Vectors are said to be like when they have the same direction and unlike when they have opposite direction.

Definition 6.8. Vectors having the same or parallel supports are called collinear vectors.

Definition 6.9. Vectors having same initial point are called coinitial vectors.

Definition 6.10. Vectors having the same terminal point are called coterminous vectors.

Definition 6.11. A vector which is drawn parallel to a given vector through a specified point in space is called localized vector.

Definition 6.12. A system of vectors is said to be coplanar, if their supports are parallel to the same plane. Otherwise they are called non-coplanar vectors.

Remark 6.2. Vectors can exist in general n-dimensional space. The general notation for a n-dimensional vector is, $\vec{v} = (a_1, a_2, a_3, \ldots a_n)$ and each of the (a_i)'s are called components of the vector.

Definition 6.13. Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be any two vectors. The addition of the two vectors is given by the following formula $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

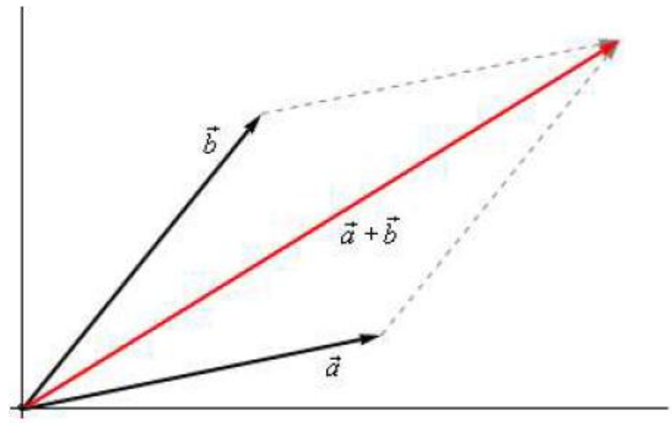

Figure 6.3. The geometric interpretation of the addition of two vectors.

Remark 6.3. This is sometimes called the parallelogram law or triangle law.

Properties of vector addition

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutativity)
- $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ (associativity)
- $\vec{a} + \vec{0} = \vec{a}$ (additive identity)
- $\vec{a} + \overline{-a} = 0$ (additive inverse)
- $(k_1 + k_2)\vec{a} = k_1\vec{a} + k_2\vec{a}$ (multiplication by scalars)
- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$ (multiplication by scalars)
- $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$ and $|\vec{a} - \vec{b}| \geq |\vec{a}| - |\vec{b}|$

Definition 6.14. Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be any two vectors. The difference (subtraction) of the two vectors is given by the following formula

$$\vec{a} + \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

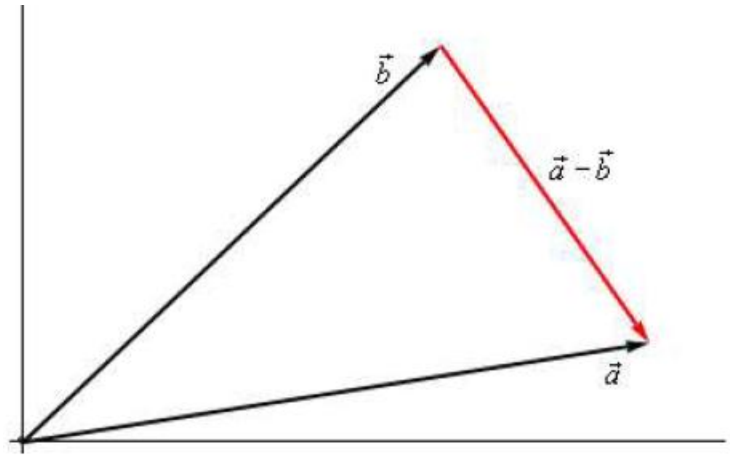

Figure 6.4. The geometric interpretation of the difference of two vectors.

It is a little harder to see this geometric interpretation. To help see this let's instead think of subtraction as the addition of $\vec{a}$ and $\overrightarrow{-b}$. First, as we'll see in a bit $\overrightarrow{-b}$ is the same vector as $\vec{b}$ with opposite signs on all the components. In other words, $\overrightarrow{-b}$ goes in the opposite direction as $\vec{b}$. Here is the vector set up for $\vec{a}+\left(\overrightarrow{-b}\right)$.

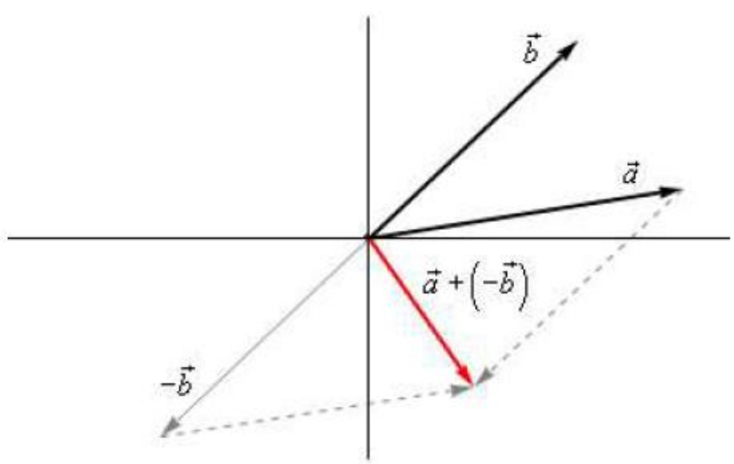

Figure 6.5. The geometric interpretation of the subtraction as the addition of two vectors.

Definition 6.15. Let $\vec{a}=(a_1,a_2,a_3)$ be a given vector and λ be a scalar. Then, the product of the vector $\vec{a}$ by the scalar λ is

$$\lambda\vec{a}=(\lambda a_1,\lambda a_2,\lambda a_3)$$

and is called the multiplication of vector by the scalar.

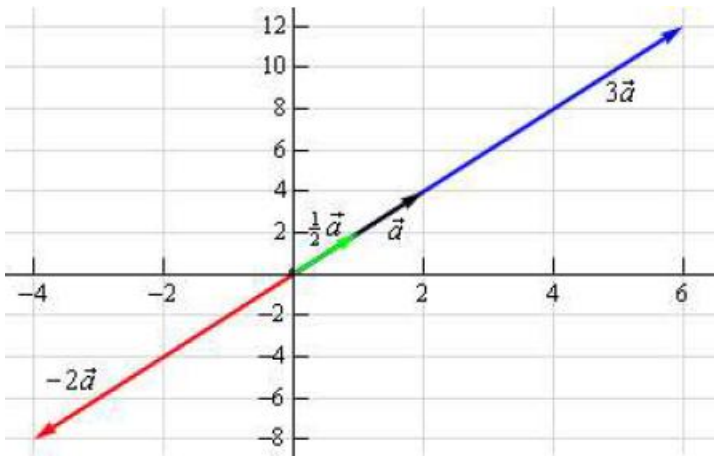

Figure 6.6. The geometric interpretation of the multiplication of vector by the scalar.

If λ is positive all scalar multiplication will do is stretch (if $\lambda > 1$) or shrink (if $\lambda < 1$) the original vector, but it won't change the direction. Likewise, if λ is negative scalar multiplication will switch the direction so that the vector will point in exactly the opposite direction and it will again stretch or shrink the magnitude of the vector depending upon the size of λ.

There are several applications of scalar multiplication. The first is parallel vectors. Two vectors are parallel if they have the same direction or are in exactly opposite directions.

Let's suppose that $\vec{a}$ and $\vec{b}$ are parallel vectors. If they are parallel then there must be a number λ so that,

$$\vec{a} = \lambda\vec{b}$$

So, two vectors are parallel if one is a scalar multiple of the other.

Important properties

- $|\lambda\vec{a}| = |\lambda||\vec{a}|$
- $|\lambda\vec{0}| = \vec{0}$
- $l(\overrightarrow{-a}) = -l\vec{a} = -(l\vec{a})$
- $(-l)(\overrightarrow{-a}) = l\vec{a}$
- $l_1(l_2\vec{a}) = l_1 l_2\vec{a} = l_2(l_1\vec{a})$
- $(l_1 + l_2)\vec{a} = l_1\vec{a} + l_2\vec{a}$

- $l(\vec{a}+\vec{b})=l\vec{a}+l\vec{b}$

Above we introduced the idea of standard basis vectors. Let's vector

$$\vec{a}=(a_1,a_2,a_3)$$

We can use the addition of vectors to break this up as follows,

$$\begin{aligned}\vec{a}\ &=(a_1,a_2,a_3)\\ &=(a_1,0,0)+(0,a_2,0)+(0,0,a_3)\end{aligned}$$

Using scalar multiplication we can further rewrite the vector as,

$$\begin{aligned}\vec{a}\ &=(a_1,0,0)+(0,a_2,0)+(0,0,a_3)\\ &=a_1(0,0,0)+a_2(0,1,0)+a_3(0,0,1)\end{aligned}$$

Finally, notice that these three new vectors are simply the three standard basis vectors for three dimensional space.

$$(a_1,a_2,a_3)=a_1\vec{\imath}+a_2\vec{\jmath}+a_3\vec{k}$$

Definition 6.16. Let $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)$ be any two vectors. The dot product is, $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$

Remark 6.4. Sometimes the dot product is called the scalar product. The dot product is also an example of an inner product and so on occasion you may hear it called an inner product.

- ## Properties of the scalar product

Let $\vec{a}=(a_1,a_2,a_3),\vec{b}=(b_1,b_2,b_3)$ and $\vec{c}=(c_1,c_2,c_3)$ be any three vectors. Then

- $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
- $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
- $\vec{a}\cdot\vec{a}=\parallel\vec{a}\parallel^2$
- $(\lambda\vec{a})\cdot\vec{b}=\vec{a}\cdot(\lambda\vec{b})=\lambda(\vec{a}\cdot\vec{b})$
- $\vec{a}\cdot\vec{0}=0$
- If $\vec{a}\cdot\vec{a}=0$ then $\vec{a}=\vec{0}$

There is also a nice geometric interpretation to the dot product. First suppose that θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \leq 0\theta 0 \leq 0\pi$ as shown in the image below.

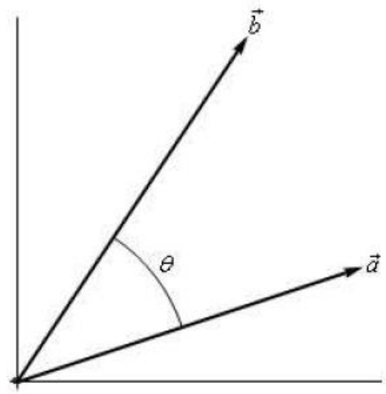

Figure 6.7.

We can then have the following theorem.

Theorem 6.1. $\vec{a} \cdot \vec{b} = \parallel \vec{a} \parallel \parallel \vec{b} \parallel cos\theta$

Remark 6.5. The formula from this theorem is often used not to compute a dot product but instead to find the angle between two vectors.

The dot product gives us a very nice method for determining if two vectors are perpendicular. In practise, we often use the term orthogonal in place of perpendicular.

If two vectors are orthogonal then we know that the angle between them is $90°$. This tells us that if two vectors are orthogonal then,

$$\vec{a} \cdot \vec{b} = 0$$

Likewise, if two vectors are parallel then the angle between them is either $0°$ (pointing in the same direction) or $180°$ (pointing in the opposite direction).

- ## Direction Cosines

Let's suppose the vector,$\vec{a}$, in three dimensional space. This vector will form angles with the x-axis (α), the y-axis (β) and the z-axis (γ). These angles are called direction angles and the cosines of these angles are called direction cosines.

Theorem 6.2. *The formulas for the direction cosines are,*

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{\|\vec{a}\|} = \frac{a_1}{\|\vec{a}\|}, \qquad \cos\beta = \frac{\vec{a}\cdot\vec{j}}{\|\vec{a}\|} = \frac{a_2}{\|\vec{a}\|}, \qquad \cos\gamma = \frac{\vec{a}\cdot\vec{k}}{\|\vec{a}\|} = \frac{a_3}{\|\vec{a}\|}$$

Remark 6.6. A couple of facts about the direction cosines

- *The vector* $\vec{u} = (\cos\alpha, \cos\beta, \cos\gamma)$ *is a unit vector.*
- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
- $\vec{a} = \|\vec{a}\| (\cos\alpha, \cos\beta, \cos\gamma)$

- ## Cross Product

Definition 6.17. Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be any two vectors then the cross product is given by the formula,

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

This is not an easy formula to remember. There are two ways to derive this formula. Both of them use the fact that the cross product is really the determinant of a 3×3 matrix. The notation for the determinant is as follows,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The first row is the standard basis vectors and must appear in the order given here. The second row is the components of $\vec{a}$ and the third row is the components of $\vec{b}$. Now, let's take a look at the different methods for getting the formula.

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

This formula is not as difficult to remember as it might at first appear to be. First, the terms alternate in sign and notice that the 2×2 is missing the column below the standard basis vector that multiplies it as well as the row of standard basis vectors.

Remark 6.7. Notice that switching the order of the vectors in the cross product simply changed all the signs in the result. Note as well that this means that the two cross products will point in exactly opposite directions since they only differ by a sign.

There is also a geometric interpretation of the cross product. We will let θ be the angle between the two vectors $\vec{a}$ and $\vec{b}$ and assume that $0 \leq \theta \leq \pi$, then we have the following fact,

$$\| \vec{a} \times \vec{b} \| = \| \vec{a} \| \, \| \vec{b} \| \, sin\theta$$

and the following figure.

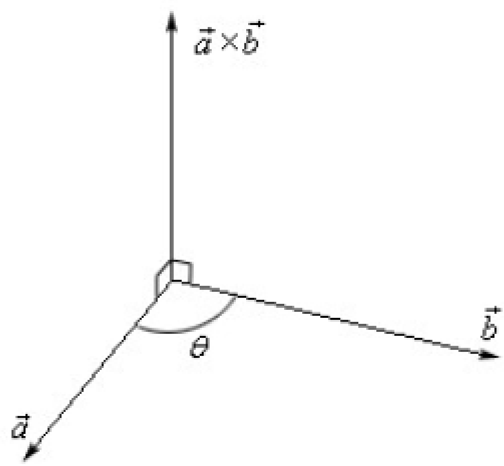

Figure 6.9.

As this figure implies, the cross product is orthogonal to both of the original vectors. This will always be the case with one exception that we'll get to in a second.

We knew that it pointed in the upward direction (in this case) by the "right hand rule". This says that if we take our right hand, start at $\vec{a}$ and rotate our fingers towards $\vec{b}$ our thumb will point in the direction of the cross product. Therefore, if we'd sketched in $\vec{b} \times \vec{a}$ above we would have gotten a vector in the downward direction.

If $\vec{a} \times \vec{b} = 0$ then $\vec{a}$ and $\vec{b}$ will be parallel vectors.

If $\vec{a} \times \vec{b} \neq 0$ then $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

- Properties

Let $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$ are vectors and λ is a number then,

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$

- $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Suppose we have three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ and we form the three dimensional figure.

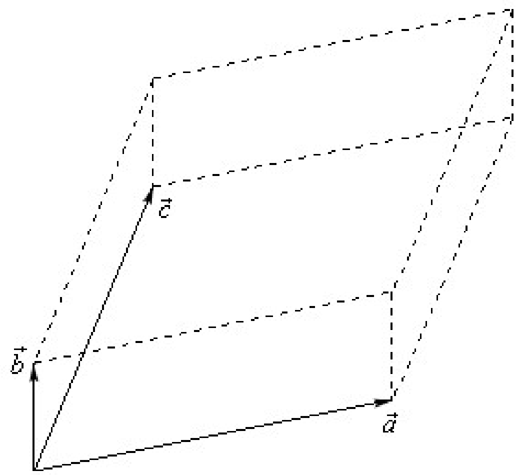

Figure 6.10.

The area S of the parallelogram (two dimensional front of this object) is given by,

$$S = \| \vec{a} \times \vec{b} \|$$

and the volume V of the parallelepiped (the whole three dimensional object) is given by,

$$V = | \vec{a} \cdot (\vec{b} \times \vec{c}) |$$

Remark 6.8. Note that the absolute value bars are required since the quantity could be negative and volume isn't negative.

We can use this volume fact to determine if three vectors lie in the same plane or not. If three vectors lie in the same plane then the volume of the parallelepiped will be zero.

5. Applications to everyday life

Vectors are utilised in day-to-day life to assist in the localization of people, places, and things. They are also used to describe things that are acting in response to an external force being applied to them. For instance, the mass of the tyre of a car that is travelling In addition to this, it possesses an initial velocity, a final velocity, and acceleration, a gravitational reaction, the forces of friction, and due to its spin, it possesses torque. A quantity that possesses both a magnitude and a direction is known as a vector. The first, second, and third laws of Newton are all relationships between vectors that precisely describe the motion of bodies when they are subjected to the influence of an outside force. Newton's laws cover a wide range of phenomena and can be used to describe anything from a ball in free fall to a rocket on its way to the moon.

- military usage
- projectile
- vector in gaming
- roller coaster
- boat crossing a river
- playing cricket
- crosswind

• Military usage

Any piece of artillery that fires a projectile by employing gun power or any other type of typically explosive-based propellant is considered to be a cannon. The calibre, range, mobility, rate of fire, and angle of fire of cannons all differ from one another. Depending on the role that each type of cannon is supposed to play on the battlefield, different types of cannon mix and balance these characteristics to differing degrees. It is required to make use of this vector. Vectors decide where the projectile will head and hit on the ground.

• Projectile

The baseball vector is utilized automatically by players in sports like basketball and baseball. In the end, students either shot the target or threw the ball in a direction at an angle, both of which were accomplished by using their understanding of vectors. In fact, in games like Javelin throw, it is necessary to judge the angle that the projectile vector makes with the ground so that the javelin can travel as far as possible.

• Vector in gaming

Vectors are utilized in the storage of locations, directions, and velocities in video games. The position vector tells us how far away the object is, the velocity vector tells us how long time it will take or how much force we need to apply, and the direction vector tells us how we should apply that force.

• Roller coaster

The majority of the motion that occurs during a roller coaster ride is a reaction to the gravitational pull that the earth exerts. It is essential to the construction of the safety system that vectors of forces, acceleration, and velocity be considered. This employs the use of vectors in designing the roller coaster ride.

- Boat crossing a river

When a boat travels over a river, the speed of the boat and the speed of the river both contribute to the total speed of the boat. When the current speed of the river changes, so does the course that the boat takes. Therefore, the boatman must determine an angle for crossing the river in order to access the shore of the river in a direct manner. Vector plays an important role here.

- Playing cricket

When the batsman hits the ball, the angle at which he shoots and the amount of velocity with which he hits the ball are both important factors in determining whether or not the ball goes over the boundary. If both of these factors meet the requirements for maximum force, the ball goes over the boundary. Because of the ball, everything revolves around the vector.

- Crosswind

The concept of a crosswind is one that is familiar to us. A wind that blows in a direction that is perpendicular to the path that one is travelling is referred to as a crosswind. When a plane finally touches down, it will at times experience challenges due to the crosswind. With the use of a vector, a pilot is able to determine the resultant velocity as well as the direction.

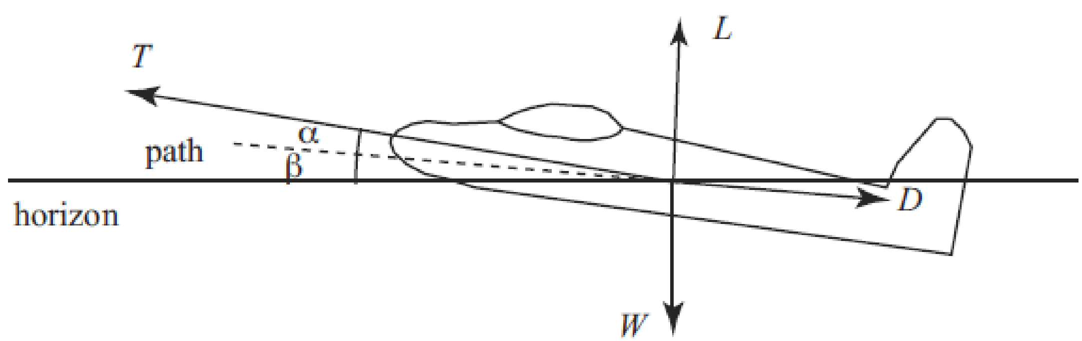

Figure 6.11. The forces acting on an aeroplane.

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7. TOPIC: Polar, cylindrical and spherical coordinates

1. Justification for topic choice

As with three dimensional space the standard (x, y, z) coordinate system is called the Cartesian coordinate system. Since this coordinate system is not always the easiest coordinate system to work in, we'll be looking at some alternate coordinate systems.

Within this topic there are introduced the polar, the cylindrical and the spherical coordinates systems.

Cylindrical coordinates are a three-dimensional coordinate system that expresses points in terms of distance from a fixed axis (z −axis), an angle around that axis (azimuth), and the height along the axis. Spherical coordinates expresses points in terms of distance from a fixed point (origin), an angle from a fixed direction (zenith), and an azimuthal angle around that direction.

In many cases, calculations involving cylindrical or spherical coordinates can be more efficient than those in Cartesian coordinates. Cylindrical coordinates are intuitive for rotational symmetry - they align with our natural understanding of rotation and height. While Cartesian coordinates are often used in everyday life due to their simplicity, cylindrical and spherical coordinates offer a valuable tool for understanding and analyzing phenomena involving cylindrical or spherical symmetry, rotational motion or spatial orientation. Their applications span a wide range of fields, from engineering and navigation to physics, engineering and robotics.

2. Historical background

The historical background of cylindrical and spherical coordinates is intertwined with the development of astronomy, cartography, and navigation. These coordinate systems have evolved over centuries, becoming indispensable tools for understanding and representing spatial relationships in various fields. The development of cylindrical and spherical coordinates can be traced back to ancient civilizations. These coordinate systems likely emerged as tools for understanding and measuring spatial relationships in astronomy and navigation. Origins of cylindrical coordinates traced back to ancient Greek mathematicians Apollonius of Perga and Hipparchus. They applied the principles of cylindrical coordinates to study lunar motion and to map out the starry night sky because they provide a convenient way to represent the position of a star relative to a celestial sphere.

The earliest known use of spherical coordinates can be traced back to the ancient Greeks. Astronomers like Aristarchus of Samos and Hipparchus employed spherical coordinates to map the positions of celestial bodies. Their work laid the foundation for the development of spherical trigonometry, which is essential for calculations involving spherical coordinates. During the middle ages and renaissance, spherical coordinates continued to be used primarily in astronomy. Astronomers like Ptolemy and Nicolaus Copernicus utilized these coordinates to develop models of the solar system. The invention of the astrolabe, a navigational instrument that relied on spherical coordinates, further solidified their importance in practical applications. Spherical coordinates were used by early sailors to determine their position on the Earth's surface. Latitude and longitude correspond to the inclination and azimuth. The 17th century saw significant advancements in the understanding of this coordinate system. Isaac Newton and Gottfried Wilhelm Leibniz developed calculus, which provided a powerful mathematical framework for analyzing spherical coordinate systems. The subsequent development of differential geometry and vector calculus further refined the theory.

3. Learning outcomes

On completion this module students should be able to

- express points and functions in polar, cylindrical and spherical coordinates,
- use the conversion formulas to convert point coordinates and domains from the Cartesian to polar, cylindrical and spherical coordinates and vice versa.

Prerequisites: Before starting this module students should

- be familiar with the concept of points and functions of two and three variables in Cartesian coordinates.

4. Theoretical foundations

• Polar coordinates

We will start with the two dimensional space and we will introduce the polar coordinate system. Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system at point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically. This is not, however, the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle φ this line makes with the positive x −axis. We could then use the distance r of the point from the origin and the amount we needed to rotate from the positive x −axis as the coordinates of the point. This is shown in the sketch below.

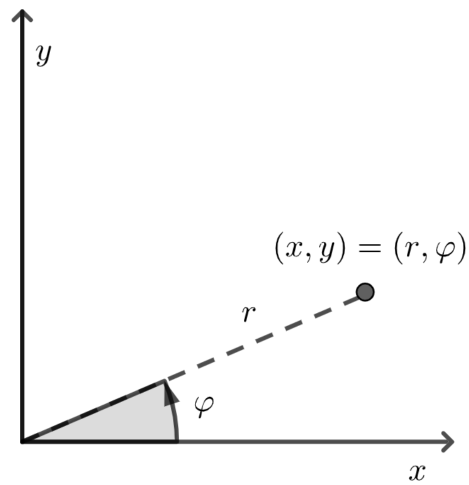

Figure 7.1.

Coordinates in this form are called **polar coordinates**.

There is an important difference between Cartesian coordinates and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.

$$\left(4,\frac{\pi}{3}\right)=\left(4,-\frac{5\pi}{3}\right)=\left(-4,\frac{4\pi}{3}\right)=\left(-4,-\frac{2\pi}{3}\right).$$

These four points only represent the coordinates of the point without rotating around the system more than once. If we allow the angle to make as many complete rotations about the axis system as we want then there are an infinite number of coordinates for the same point. In fact, the point (r,φ) can be represented by any of the following coordinate pairs:

$$(r,\varphi+2\pi n),\qquad(-r,\varphi+(2n+1)\pi),\ \text{where } n\in\mathbb{Z}.$$

We need to think about converting between the two coordinate systems. Using the right triangle above we can get the following polar to Cartesian conversion formulae:

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi. \end{aligned}$$

For converting from Cartesian, let us notice this very useful formula

$$x^2+y^2=(r\cos\varphi)^2+(r\sin\varphi)^2=r^2\,(\cos^2\varphi+\sin^2\varphi)=r^2\cdot 1=r^2.$$

Taking the square root of both sides (with the convention of positive r), we have $r=\sqrt{x^2+y^2}$.

To get the equation for φ, we start with $\frac{y}{x} = \frac{r\sin\varphi}{r\cos\varphi} = tan\varphi$ and taking the inverse tangent of both sides gives, $\varphi = arctan\left(\frac{y}{x}\right)$. We will need to be careful with this because inverse tangents only return values in the range $-\pi/2 < \varphi < \pi/2$. Recall that there is a second possible angle and that the second angle is given by $\varphi + \pi$.

To summarize this we have the Cartesian to polar conversion formulae:

$$r = \sqrt{x^2 + y^2} \text{ or } r^2 = x^2 + y^2$$

$$\varphi = \arctan\left(\frac{y}{x}\right).$$

- ## Cylindrical coordinates

Cylindrical coordinate system is fairly simple as it is nothing more than an extension of polar coordinates into three dimensions. All that is done is to add a z on as the third coordinate. The r and φ are the same as with polar coordinates. Here is a sketch of a point in $\mathbb{R}^3$.

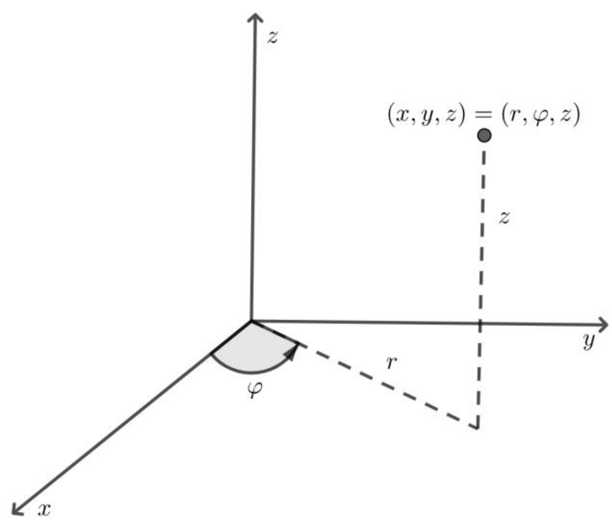

Figure 7.2.

The conversions for x and y are the same conversions that is used back when looking at polar coordinates. So, if there is a point in cylindrical coordinates the Cartesian coordinates can be found by using the following conversions:

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z. \end{aligned}$$

The third equation is just an acknowledgement that the z −coordinate of a point in Cartesian and polar coordinates is the same.

Likewise, if we have a point in Cartesian coordinates the cylindrical coordinates can be found by using the following conversions:

$$\begin{aligned} r &= \sqrt{x^2+y^2} \text{ or } r^2 = x^2+y^2 \\ \varphi &= \arctan\left(\frac{y}{x}\right) \\ z &= z. \end{aligned}$$

- Spherical coordinates

Spherical coordinate system consists of the following three quantities. First there is ρ. This is the distance from the origin to the point and there is required $\rho \geq 0$. Next there is φ. This is the same angle that we saw in polar / cylindrical coordinates. It is the angle between the positive x −axis and the line above denoted by r (which is also the same r as in polar/cylindrical coordinates). There are no restrictions on φ. Finally, there is θ. This is the angle between the positive z −axis and the line from the origin to the point. It is required $0 \leq \theta \leq \pi$. In summary, ρ is the distance from the origin to the point, θ is the angle that we need to rotate down from the positive z −axis to get to the point and φ is how much we need to rotate around the z −axis to get to the point. Here is a sketch of a point in $\mathbb{R}^3$.

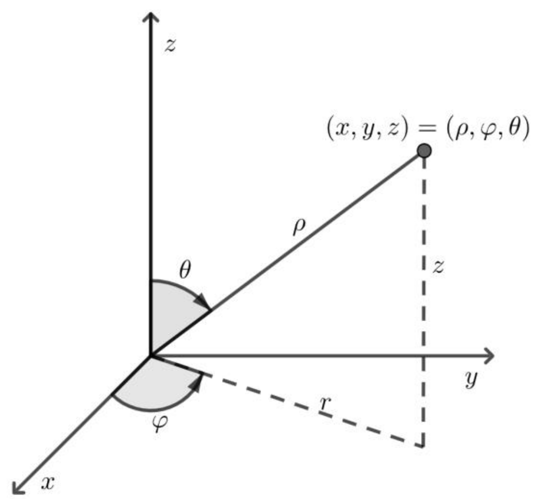

Figure 7.3.

To derive the conversion formulas we first start with a point in spherical coordinates and ask what the cylindrical coordinates of the point are. So, (ρ, φ, θ) are known and (r, φ, z) should be found. In fact, we only need to find ρ and z since φ is the same in both coordinate systems. If we look at the sketch above from directly in front of the triangle we get the following sketch:

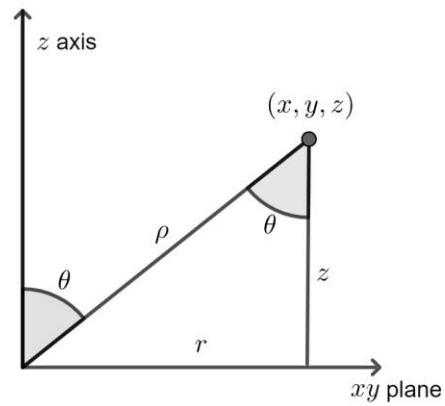

Figure 7.4.

We know that the angle between the z −axis and ρ is θ and with a little geometry we also know that the angle between ρ and the vertical side of the right triangle is also θ. Then, with a little right triangle trig we get,

$$\begin{aligned} z &= \rho\cos\theta \\ r &= \rho\sin\theta. \end{aligned}$$

and these are exactly the formulas that we were looking for. So, given a point in spherical coordinates the cylindrical coordinates of the point will be:

$$\begin{aligned} r &= \rho\sin\theta \\ \varphi &= \varphi \\ z &= \rho\cos\theta. \end{aligned}$$

Note as well from the Pythagorean theorem we also get, $\rho^2 = r^2 + z^2$.

Next, let's find the Cartesian coordinates of the same point. To do this we'll start with the cylindrical conversion formulas from the previous section, and all that we need to do is to use the formulas from above for r and z to get:

$$\begin{aligned} x &= \rho\sin\theta\cos\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}$$

Also note that since we know that $r^2 = x^2 + y^2$ we get, $\rho^2 = x^2 + y^2 + z^2$. Converting points from Cartesian or cylindrical coordinates into spherical coordinates is usually done with the same conversion formulas.

5. Applications to everyday life

In the modern era, cylindrical and spherical coordinates have become essential tools in various fields, including physics, engineering, and computer graphics. They are used to describe the motion of particles, to model physical phenomena, and to represent three-dimensional data. More specifically, cylindrical coordinates are used in fluid dynamics to calculate flow rates in pipes or the impact of forces on submerged surfaces, in structural engineering when dealing with structures consisting of circular components like towers, cylinders, or domes, and, in software engineering, especially in 3D graphics and simulations where the circular or cylindrical traits of an object need to be represented digitally.

Cylindrical coordinates are often used to describe the geometry of cylindrical objects like pipes, shafts, or gears. This makes it easier to design and analyze their properties. In manufacturing processes like lathe turning or milling, cylindrical coordinates are used to control the movement of tools to create cylindrical shapes.

Fields like electric and magnetic fields can be described more conveniently using cylindrical coordinates, especially when dealing with cylindrical symmetry, such as long wires or solenoids, and using spherical coordinates, especially when dealing with symmetric situations like charges or currents.

Cylindrical and spherical coordinates are used to analyze the flow of fluids around cylindrical objects, such as pipes or columns, or around objects with spherical symmetry, such as bubbles or droplets.

In problems involving heat transfer, cylindrical coordinates can be useful for analyzing situations with cylindrical symmetry, like heat conduction in a pipe or a cylinder.

Robotic arms often use spherical coordinates to specify the position and orientation of their end-effector. This allows for precise movements and tasks like welding or assembly.

Sensor Data Processing: Sensors on robots or autonomous vehicles might use cylindrical or spherical coordinates to represent data like the direction of sound or the position of objects, in case of cylindrical coordinates objects are relative to a central axis.

Meteorologists use cylindrical coordinates to describe wind patterns around a central point, helping to understand and predict weather conditions.

GPS systems use spherical coordinates to pinpoint the exact location of a device on the Earth’s surface. The latitude and longitude are essentially spherical coordinates, where latitude represents the zenith angle and longitude represents the azimuthal angle.

To determine the distance between two points, spherical coordinates are used in conjunction with the Earth’s radius to calculate the shortest path between them.

In virtual reality, objects are often modeled using spherical coordinates. This allows for accurate representation and manipulation of three-dimensional objects.

Astronomers use spherical coordinates to track the positions of stars, planets, and other celestial bodies. The right ascension and declination are equivalent to the azimuthal and zenith angles, respectively.

Understanding the orbits of celestial bodies involves using spherical coordinates to describe their position and velocity relative to a central point.

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8. TOPIC: Trigonometric functions

1. Justification for topic choice

Trigonometry is a critical branch of mathematics, forming the foundation for advanced topics in calculus, physics, engineering, and architecture. It is essential for students to master trigonometric concepts as they progress in their academic and professional careers. However, trigonometry can be abstract and challenging for many learners, particularly when it comes to visualizing and understanding the spatial relationships involved.

Using VR in the study of trigonometry addresses these challenges by providing a fully immersive learning experience. VR enables students to visualize trigonometric concepts in three dimensions, making it easier to grasp complex ideas. For example, students can manipulate angles and triangles in a virtual space, observe the effects of changes in real-time, and better understand the unit circle and its applications. This interactive, visual approach helps to reinforce learning, making trigonometry more accessible and engaging for students with different learning styles.

This module leverages virtual reality (VR) to teach the fundamental concepts of trigonometry. Students will engage with 3D environments to explore and understand the relationships between angles and sides in right-angled triangles, the unit circle, and trigonometric functions like sine, cosine, and tangent. The VR experience allows for an interactive exploration of these concepts, providing a hands-on approach to understanding the principles of trigonometry.

2. Historical background

Trigonometry (from Ancient Greek $\tau\rho\iota\gamma\omega\nu o\nu$ (trígōnon) 'triangle' and $\mu\varepsilon\tau\rho o\nu$ (métron) 'measure') is a branch of mathematics concerned with relationships between angles and side lengths of triangles. In particular, the trigonometric functions relate the angles of a right triangle with ratios of its side lengths.

Trigonometry has a long and rich history, dating back to ancient civilizations. It emerged in the Hellenistic world during the 3rd century BC from applications of geometry to astronomical studies. Early study of triangles can be traced to the 2nd millennium BC, in Egyptian mathematics (*Rhind Mathematical Papyrus*) and Babylonian mathematics. The Babylonians and Egyptians used early trigonometric ideas for practical purposes like astronomy and land surveying. Trigonometry was also prevalent in Kushite mathematics.

The formal study of trigonometry began with the Greeks, particularly with the work of *Hipparchus* around 150 BCE, who is credited with compiling the first known trigonometric table. *Ptolemy* later expanded on Hipparchus's work, contributing significantly to the field. In Indian astronomy, the study of trigonometric functions flourished in the Gupta period, especially due to

Aryabhata (sixth century AD), who discovered the sine, cosine, and other trigonometric functions.

The Greeks focused on the calculation of chords, while mathematicians in India created the earliest-known tables of values for trigonometric ratios (trigonometric functions).

Trigonometry became an independent discipline in the Islamic world, where all six trigonometric functions were known. During the Islamic Golden Age, scholars such as *Al-Battani*, *Abu al-Wafa* and *Al-Khwarizmi* made further advancements, particularly in spherical trigonometry, which was vital for navigation and astronomy. Translations of Arabic and Greek texts led to trigonometry being adopted as a subject in the Latin West beginning in the Renaissance with Regiomontanus.

The development of modern trigonometry shifted during the western Age of Enlightenment, beginning with 17th-century mathematics (*Johannes Kepler*, *Isaac Newton*, *James Stirling* et al.) and reaching its modern form with *Leonhard Euler* (1748). Their work helped integrate trigonometry into the broader field of mathematics and its applications in science.

By incorporating VR into the study of trigonometry, this module connects students with both the historical evolution of the subject and the cutting-edge technology of today. This approach not only enhances the learning experience but also illustrates the continued relevance and application of trigonometry in the modern world.

3. Learning outcomes

Understanding of Basic Trigonometric Concepts:

- Comprehension of Trigonometric Ratios: Students should be able to define and explain the basic trigonometric ratios—sine, cosine, and tangent—in the context of a right-angled triangle.
- Application of Ratios: Students should be able to apply these ratios to solve simple problems involving the sides and angles of right-angled triangles.

Visualization of Trigonometric Relationships:

- Unit Circle Exploration: Students should gain a basic understanding of the unit circle, including how angles correspond to coordinates on the circle and how this relates to the sine and cosine functions.
- Real-Time Interaction: Using VR, students should be able to visualize how changing an angle in a triangle or on the unit circle affects the sine, cosine, and tangent values, enhancing their spatial understanding of these concepts.

Graphing Trigonometric Functions:

- Basic Graph Interpretation: Students should be able to recognize and interpret the graphs of sine, cosine, and tangent functions, understanding their key characteristics like amplitude, period, and frequency.
- Graph Creation: Students should be able to sketch or identify basic graphs of these functions based on given data or VR simulations.

Real-World Application Awareness:

- Connection to Real-World Scenarios: Students should be able to identify and describe at least one real-world application of trigonometry (e.g., navigation, architecture, or physics) as demonstrated in the VR environment.

Improved Engagement and Confidence:

- Increased Comfort with Trigonometry: The immersive nature of VR should help students feel more confident and engaged with trigonometric concepts, reducing anxiety or frustration often associated with learning mathematics.
- Enhanced Problem-Solving Skills: Students should demonstrate improved problem-solving abilities by applying their understanding of trigonometric concepts to VR-based scenarios or simple word problems.

4. Theoretical foundation

1. Radian Angle Measurement

Along with the practical degree measurement of angles, the radian measurement is used in theoretical questions, the value of the angle α, which is central to the circumference, is measured by the ratio of the length l of the arc on which this angle rests to the length of the radius r of this circle: $\alpha = \frac{l}{r}$. In this measurement, the unit is *radian* – the angle that is central to an arc whose length is equal to the radius of the circle. One *radian* is equal to 57º17’44”,8, which means that 1º=0.017453 radian. The transition from one dimension to another is made according to formulas:

$$\alpha^0 = \frac{180}{\pi}\alpha(\text{rad}), \quad \alpha(rad) = \frac{\pi}{180}\alpha^0.$$

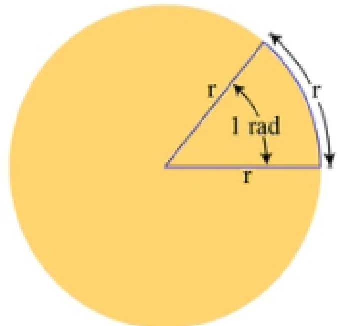

Figure 8.1.

2. Trigonometric Ratios

Trigonometric angular functions are determined using a trigonometric circle as well as from a sharp-angled triangle (for acute angles). Trigonometric circle is a circle with a radius of 1, centered at the origin of the coordinate plane. The Points 0, A, and B form the corresponding triangle.

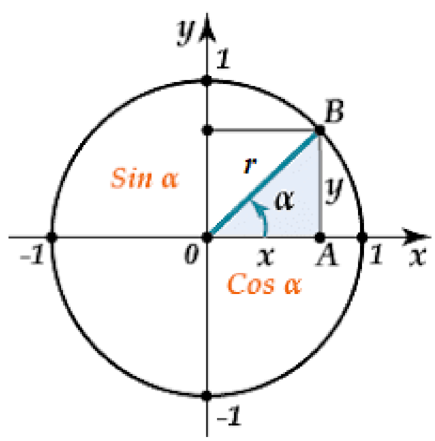

Figure 8.2.

The coordinates (x, y) correspond to ($(cosα, sinα)$) for a given angle (α) as follows:

- Sine $(\sin)$: Defined as the ratio of the opposite side to the hypotenuse in a right-angled triangle:

$$\sin\alpha = \frac{y}{r} = y = AB.$$

- Cosine $(\cos)$: Defined as the ratio of the adjacent side to the hypotenuse:

$$\cos\alpha = \frac{x}{r} = x = 0B.$$

- Tangent (tan): Defined as the ratio of the opposite side to the adjacent side:

$$\tan\alpha = \frac{y}{x} = \frac{AB}{0B}.$$

Other trigonometric functions can be defined based on the previous definitions:

- Contangent (cot): Defined as tan:

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{0B}{AB}.$$

- Secante (sec): Defined as the reciprocal value to cos:

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{x}.$$

- Cosecante (csc): Defined as the reciprocal value to sin:

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{1}{y}.$$

When the angle varies from 0 to 2π (in any direction), the corresponding values of trigonometric functions are defined according to the symmetry and with the corresponding signs.

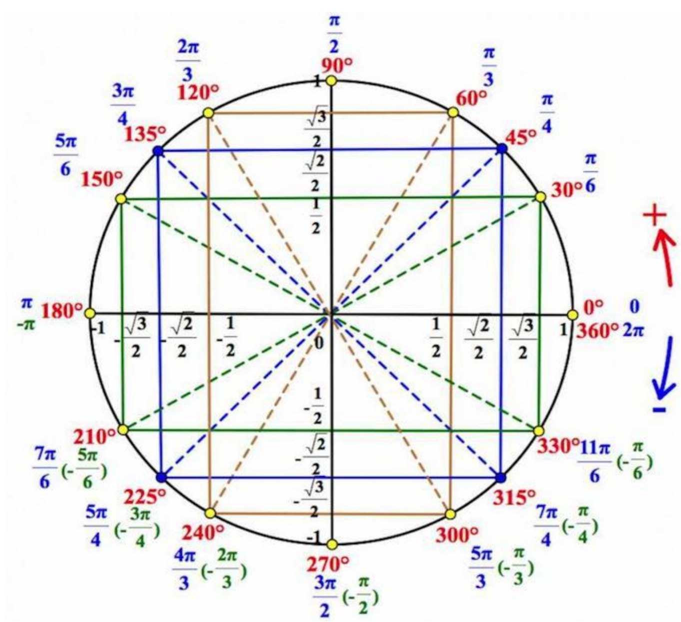

Figure 8.3.

From the definitions, we can easily conclude that

- the trigonometric functions $f(x) = \sin x$, $f(x) = \cos x$ are defined in $\mathbb{R}$;
- the trigonometric functions $f(x) = \tan x$ and $f(x) = sec x$, are defined in $\mathbb{R}\{\frac{k\pi}{2}, \forall k \in \mathbb{Z}\}$;
- the trigonometric functions $f(x) = \cot x$ and $f(x) = \csc x$, are defined in $\mathbb{R}\{k\pi, \forall k \in \mathbb{Z}\}$;
- the trigonometric functions $f(x) = \sin x$, $f(x) = \cos x$, $f(x) = \sec x$, and $f(x) = \csc x$ are periodic with period 2π: $f(x) = f(x + 2k\pi), \forall k \in \mathbb{Z}$, *i.e.*

$$\sin(2k\pi + x) = \sin x,\ \cos(2k\pi + x) = \cos x,$$
$$\sec(2k\pi + x) = \sec x,\ \csc(2k\pi + x) = \csc x$$

- the trigonometric functions *tan* and *cot* are periodic with period π: $f(x) = f(x + k\pi), \forall k \in \mathbb{Z}$*i.e.*

$$\tan\left(\frac{k\pi}{2} + x\right) = \tan x,\ \cot\left(\frac{k\pi}{2} + x\right) = \cot x.$$

- the trigonometric functions *cos* and *sec* are even= $f(-x) = f(x), \forall x \in D_f$;
- the trigonometric functions *sin*, *csc*, *tan*, and *cot* are odd: $f(-x) = -f(x), \forall x \in D_f$;
- for all $x \in \mathbb{R}$, we have

$$-1 \leq \sin x \leq 1;\ -1 \leq \cos x \leq 1;$$

- for all $x \in D_f$, the values of $\tan x$ and $\cot x$, vary from $-\infty$ to $+\infty$;
- for all $x \in D_f$, the values of $\sec x$ and $\csc x$, vary in $]-\infty, -1[\cup\ \ ]1, +\infty[$.

Basic Trigonometric Identities:

- Pythagorean Identity: $\sin^2 x + \cos^2 x = 1$;
- $\sin x \csc x = 1,\ \cos x \sec x = 1$;
- $\sec^2 \pm \tan^2 x = 1,\ \csc^2 x - \cot^2 x = 1.$

Representation of one trigonometric function in terms of another:

- $\sin x = \sqrt{1 - \cos^2 x} = \frac{\tan x}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{1+\cot^2 x}}$;
- $\cos x = \sqrt{1 - \sin^2 x} = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{\cot x}{\sqrt{1+\cot^2 x}}$;
- $\tan x = \frac{\sin x}{\sqrt{1-\sin^2 x}} = \frac{\sqrt{1-\cos^2 x}}{\cos x} = \sqrt{\sec^2 x - 1} = \frac{1}{\sqrt{\csc^2 x - 1}}$;

- $\cot x = \frac{\sqrt{1-\sin^2 x}}{\sin x} = \frac{\cos x}{\sqrt{1-\cos^2 x}} = \frac{1}{\sqrt{\sec^2 x - 1}} = \sqrt{\csc^2 x - 1}$.

3. **Angle sum and angle difference formulas:**

- $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$;
- $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$;
- $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$;
- $\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$.

4. **Formulas of the multiple angles:**

- $\sin(2x) = 2\sin x \cos x$;
- $\cos(2x) = \cos^2 x - \sin^2 x$;
- $\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$;
- $\cot(2x) = \frac{\cot^2 x - 1}{2\cot x}$.

5. **Half angle formulas:**

- $\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1}{2}(1-\cos x)}$;
- $\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1}{2}(1+\cos x)}$;
- $\tan\left(\frac{x}{2}\right) = \sqrt{\frac{(1-\cos x)}{(1+\cos x)}} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$;
- $\cot\left(\frac{x}{2}\right) = \sqrt{\frac{(1+\cos x)}{(1-\cos x)}} = \frac{1+\cos x}{\sin x} = \frac{\sin x}{1-\cos x}$.

6. **Sum and difference of trigonometric functions:**

- $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$;
- $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$;
- $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$;

- $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$;
- $\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}$;
- $\cot x \pm \cot y = \pm\frac{\sin(x \pm y)}{\sin x \sin y}$.

7. Products of some trigonometric functions:

- $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$;
- $\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$;
- $\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$.

8. Powers of some trigonometric functions:

- $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$; $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$.

9. Graphics of main trigonometric functions:

- Graphics of function $\sin x$:

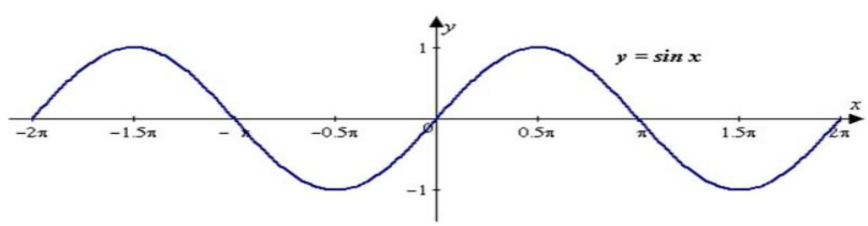

Figure 8.4.

- Graphics of function $\cos x$:

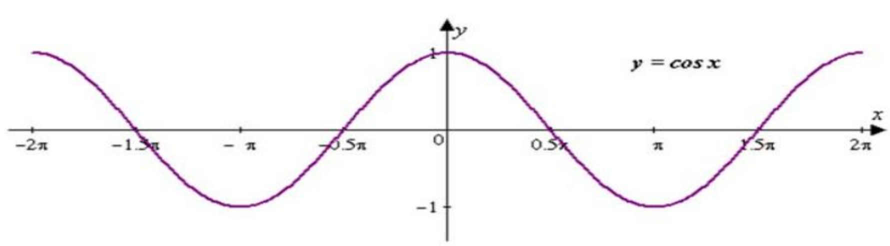

Figure 8.5.

- Graphics of function $\tan x$:

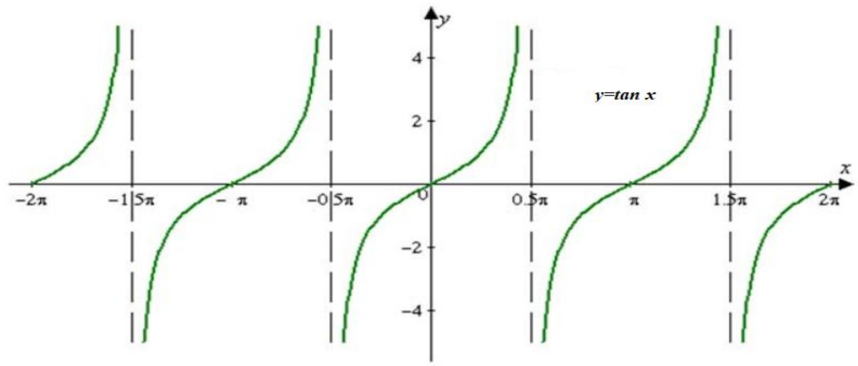

Figure 8.6.

- Graphics of function $\cot x$:

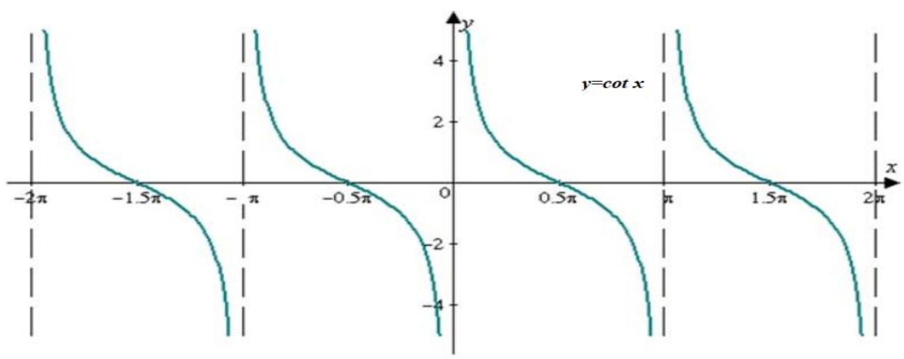

Figure 8.7.

• Examples

- Construct the graphics of function $y = tan\left(x - \frac{\pi}{4}\right)$.
  Solution. The desired graph is obtained from the graph of the function $y = \tan x$ as a result of parallel transfer along the abscissa axis to the right by $\frac{\pi}{4}$:

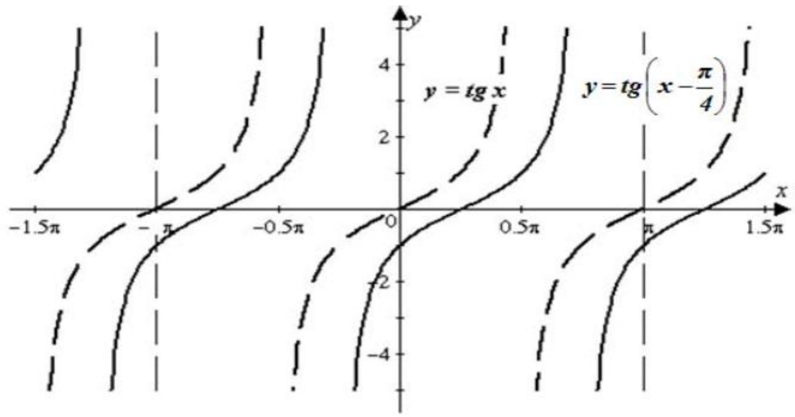

Figure 8.8.

- Construct the graphics of function $y = \sin x + 1$.
  Solution. The desired graph is obtained from the graph of the function $y = \sin x$ as a result of parallel transfer along the ordinate axis up by 1:

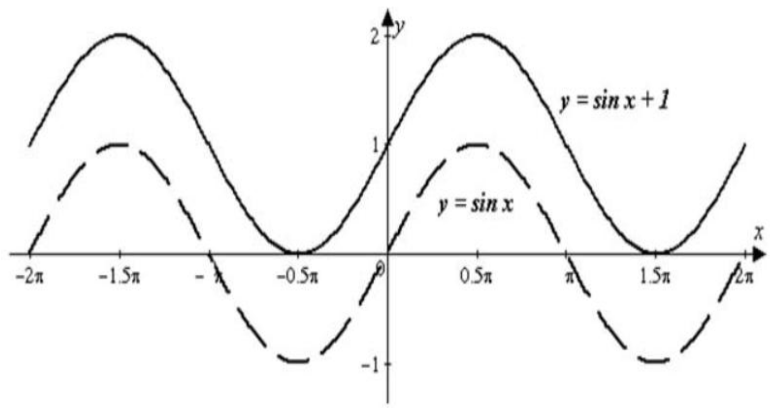

Figure 8.9.

- Construct the graphics of function $y = 3\cot x$.
  Solution. The desired graph is obtained from the graph of the function $y = \cot x$ by stretching the latter along the ordinate axis by three times (increasing the distance from each point of the graph to the axis of the abscissa by three times):

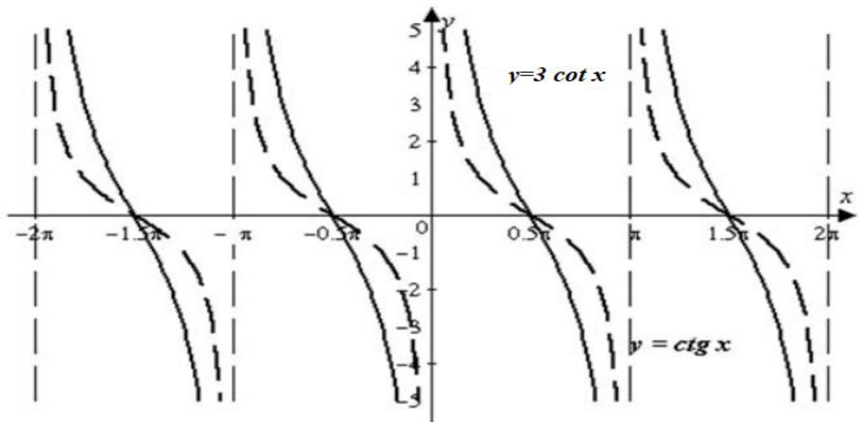

Figure 8.10.

- Construct the graphics of function $y = -\cos\left(x + \frac{\pi}{3}\right)$.
  Solution. We will construct a given graph using elementary transformations of the graph of the function $y = \cos\left(x + \frac{\pi}{3}\right)$. By carrying out a parallel transfer of this graph along the x-axis to the left by $\frac{\pi}{3}$, we get the graph of $y = \cos\left(x + \frac{\pi}{3}\right)$, we will get:

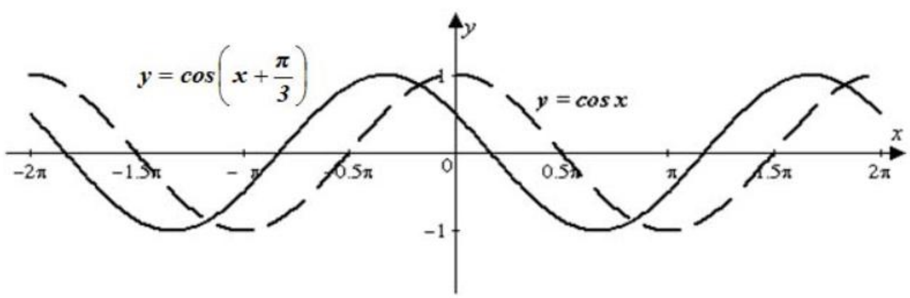

Figure 8.11.

Then, by displaying the graph of the function $y = \cos\left(x + \frac{\pi}{3}\right)$ symmetrically with respect to the abscissa axis, we get the desired graph:

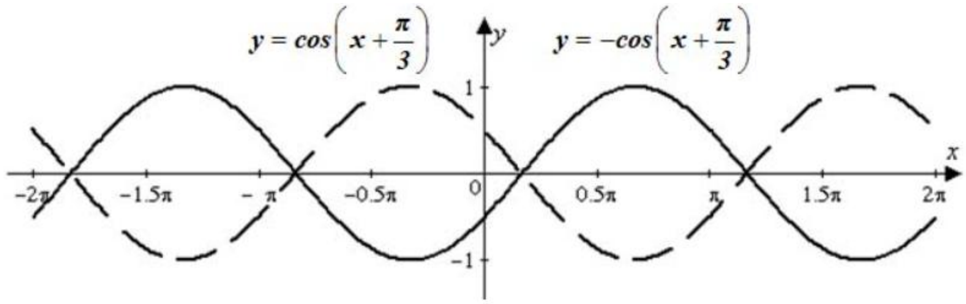

Figure 8.12.

- Introduction to Inverse Trigonometric Functions

Inverse trigonometric functions are the inverse operations of the primary trigonometric functions: sine, cosine, tangent, cotangent, secant, and cosecant. These functions, typically denoted as arcsin, arccos, arctan, arccot, arcsec, and arccsc, allow us to determine the angle that corresponds to a given trigonometric ratio.

For instance, if $sin\theta = x$ for some angle θ, then the angle can be expressed as:

$$\theta = \arcsin(x).$$

Each inverse trigonometric function has a restricted range to ensure it is well-defined and produces a unique output. These ranges are chosen so that each inverse function captures one complete cycle of the original function. The specific ranges are as follows:

- $\arcsin(x) : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\arccos(x) : [-1,1] \to [0, \pi]$
- $\arctan(x) : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The relationship between a trigonometric function and its inverse is defined by the original function's range and the inverse function's domain. For example, the sine function $sin(x)$ is defined for all real values of x, but the values of $\sin(x)$ range only between $[-1,1]$. Consequently, the inverse sine function, $\arcsin(x)$, is defined only for $x \in [-1,1]$ and produces an angle within the restricted range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

This restriction on the domain of the inverse function helps resolve ambiguities. Since trigonometric functions are periodic, they do not have a unique inverse without this restriction. By limiting the range of the output angles, each inverse function selects a principal value, ensuring a one-to-one correspondence.

Inverse trigonometric functions play a crucial role in solving equations where angles must be determined from known trigonometric values. Applications of these functions extend across various fields such as physics, engineering, and computer science, especially in contexts where angle determination is necessary.

Inverse trigonometric functions have applications in various fields, including physics, engineering, and computer graphics, where determining angles from known ratios is essential.

5. Applications to everyday life

Trigonometry may not have its direct applications in solving practical issues, but it is used in various things that we enjoy so much. For example, music, as you know sound travels in waves, and this pattern though not as regular as functions sin or cos, is still useful in developing computer music. A computer cannot obviously listen to and comprehend music as we do, so computers represent it mathematically by its constituent sound waves. And this means sound engineers need to know at least the basics of trigonometry. And the good music that these sound engineers produce is used to calm us from our hectic, stress full life – All thanks to trigonometry.

- ### Trigonometry in astronomy

People have always been attracted to space. Astronomy has been the driving force behind advances in trigonometry. Most of the early discoveries in trigonometry were related to spherical trigonometry, mainly because of its application to astronomy. The three major figures we know of in the development of Greek trigonometry are *Hipparchus*, *Menelaus*, and *Ptolemy*. There were probably other authors, but over time, their works were lost and their names were forgotten. How far are the stars? Throughout the year, the Earth revolves around the Sun, and the apparent position of the star appears to shift slightly relative to the stars that are much farther away. You can observe this phenomenon quite easily. Extend your arm in front of you with your thumb up. Then, look at it first with your left and then with your right eye. You may notice that the finger changes position. If you bring your finger closer to your eyes, the offset in relation to the background will be greater. Our eyes are located at a certain distance, which is why the straight lines that we mentally draw from the finger to the eyes create an angle. As we continue these straights, we will find two different positions of the finger. And the angle between them will depend on the closer the finger is to the eyes. This phenomenon is known as parallax. Astronomers estimate the distance to nearby objects in space using this method, called stellar parallax, or trigonometric parallax.

- ### Trigonometry in measuring the heights of a building or mountains

Trigonometry can be used to measure the height of, for example, mountains: if you know the distance from which you are observing and the angle of elevation, you can easily determine the height of a mountain. Likewise, if you have the value of one side and the angle of inclination from the top of the hill, you can find the other side in the triangle, all you need to know is the one side and the angle of the triangle. According to proven facts, American scientists say that in order to determine the distance to any object, our brain first estimates the angle between the plane of vision and the plane of the earth. In general, the idea of "measuring angles" is not new. Distant objects higher in the field of view were drawn by the painters of Ancient China, somewhat neglecting the laws of perspective. The Arab scientist of the eleventh century, *Alhazen*, formulated the theory of determining distance by estimating angles.

• Trigonometry in construction

In construction of buildings, roads, bridges, etc., people need trigonometry to:

- Measure fields, lots and areas;
- measures the height of the building, the width length etc.
- make walls parallel and perpendicular;
- install ceramic tiles;
- design roof inclination;

Architects use trigonometry to calculate structural load, roof slopes, ground surfaces and many other aspects, including sun shading and light angles.

Trigonometry can be called the genius of architecture, most of the buildings we know were designed thanks to it. Some famous examples of such buildings are:

- Mary Axe in London:

- the Sydney Opera House:

- the Restaurant in Los Manantiales in Argentina:

- the Gaudí Children's School in Barcelona:

- the Bodegas Isios Winery in Spain:

- ## Trigonometry in flight engineering

Flight engineers have to take in account their speed, distance, and direction along with the speed and direction of the wind. The wind plays an important role in how and when a plane will arrive where ever needed this is solved using vectors to create a triangle using trigonometry to solve.

- ## Trigonometry in Biology

One of the characteristic features of living nature is the cyclical nature of most of the processes occurring in it. There is a connection between the movement of celestial bodies and living organisms. Biorhythms (biological rhythms) are relatively regular changes in the nature of a living organism and the intensity of its biological processes. This is common in all living organisms, and the ability to make such changes is inherited. This phenomenon can be observed both in individual cells and in entire populations of living organisms. Biological rhythms are divided into ecological (coinciding with the rhythm of the environment) and physiological (periods from fractions of a second to several minutes). In terms of time, biorhythms can be seasonal, annual, diurnal, and so on. With the help of trigonometric functions, it is also possible to build a model of biorhythms.

Marine biologists often use trigonometry to establish measurements. For example, to find out how light levels at different depths affect the ability of algae to photosynthesize. Trigonometry is used in finding the distance between celestial bodies. Also, marine biologists utilize mathematical models to measure and understand sea animals and their behaviour. Marine biologists may use trigonometry to determine the size of wild animals from a distance.

- ## Trigonometry in navigation

Trigonometry is used to set directions such as the North, South, East, and West. It tells you what direction to take with the compass to get on a straight direction. It is used in navigation in order to pinpoint a location. It is also used to find the distance of the shore from a point in the sea. It is also used to see the horizon.

- ## Trigonometry in video games

Imagine a gamer smoothly gliding over the road blocks. He doesn’t really jump straight along the Y axis, it is a slightly curved path or a parabolic path that he takes to tackle the obstacles on his way. Trigonometry helps him to jump over these obstacles. Gaming industry is all about IT and computers and Trigonometry is of equal importance for these engineers.

6. References

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9. TOPIC: Non-euclidean geometry

1. Justification for topic choice

Non-Euclidean geometry (NEG) offers an alternative and insightful perspective on the foundations of axiomatic systems, fundamentally reshaping our understanding of geometry (Coxeter, 1965). By challenging the deeply embedded assumptions of Euclidean geometry, NEG opens up the way for the development of advanced cognitive skills essential in mathematical education and broader scientific inquiry. In engaging with non-Euclidean geometries, students actively confront the limitations of Euclidean frameworks, pushing them to analyze multiple models critically. This process strengthens their logical reasoning and critical thinking and fosters creative problem-solving skills, which are invaluable for scientific innovation (Buda, 2017; Sukestiyarno et al., 2023; Kranz et al., 2014).

Exploring non-Euclidean spaces is particularly enriching because it immerses students and teachers in situations that defy the usual Euclidean intuition, such as the geometry of curved surfaces in hyperbolic or elliptic contexts.

Euclid's fifth postulate, often restated as Playfair's postulate, claims that for any line l and any point $A \notin l$, there is one, and only one, line, s, such that $A \in s$ and s is parallel to l, however, this is not the case in non-euclidean geometries. Hyperbolic geometry, for example, allows for infinitely many parallel lines to l through A, while elliptic geometry permits none. This setting off from the Euclidean norm forces students to reevaluate what it means for space to be a *line* or a *plane,* an exercise that is mentally demanding yet incredibly rewarding in terms of intellectual growth.

NEG also emphasizes the power of the axiomatic approach, a methodology foundational to scientific progress across distinct branches of knowledge. By engaging with alternative axiom sets and seeing how small changes in foundational assumptions yield vastly different mathematical structures, students gain insight into geometry and learn to appreciate the rigor of formal methods. These experiences are invaluable in advanced studies in, for instance, topology, differential geometry, and theoretical physics (the geometry of spacetime), where these methods are applied.

In educational contexts, particularly in dynamic environments such as GeoGebra, and in VR-based projects, NEG can be visualized in ways that traditional Euclidean configurations cannot capture, allowing future teachers and students to explore these complex ideas interactively. Through projects promoting model exploration and the collaborative development of geometric understanding, students can see NEG not as an abstract or peculiar area of mathematics but as a stimulating, essential aspect of mathematical inquiry with direct implications for scientific and technological fields.

2. Historical background

For many centuries, Euclidean geometry was considered the ideal model for "the real world". However, the fifth postulate, known as the parallel postulate, became a source of debate among mathematicians. Should it be considered a postulate or a theorem? Numerous attempts were made to prove it, but none were successful.

Among the pioneers was Carl Friedrich Gauss, the "Prince of Mathematicians," who conducted significant research in non-Euclidean geometry, although he chose not to publish his results, (Coxeter, 1977). Another distinguished contributor was the Hungarian mathematician János Bolyai, who formulated a version of non-Euclidean geometry, which he termed *absolute geometry*, (Gray, 2004). Bolyai shared his discoveries with his father, Farkas Bolyai, who strongly advised him to publish his work. Independently and around the same time the Russian mathematician, Nikolai Ivanovich Lobachevsky developed a similar form of non-Euclidean geometry, which he called *imaginary geometry*, (Bonola, 1955). The work done by these mathematicians showed the feasibility of constructing coherent geometric (axiomatic) systems, taking out the parallel postulate, leading to the emergence and development of non-Euclidean geometries. The discovery of non-Euclidean geometries had far-reaching implications, challenging the long-standing belief that Euclidean geometry was the only valid representation of the space, we live in.

3. Learning outcomes

Upon completing this module, students should be able to:

- Understand how changing axioms can lead to the development of entirely new geometric structures, deepening the understanding of the nature of mathematical systems;
- Develop critical thinking skills by comparing and contrasting Euclidean and non-Euclidean geometries, promoting the ability to cross-examine established assumptions;
- Improve deductive reasoning through proofs and theorems valid only in non-Euclidean structures, allowing the development of increasingly complex logical-deductive constructive chains;
- Analyze and evaluate multiple geometric models, understanding the strengths and limitations of each of them;
- Improve spatial reasoning skills by engaging with non-intuitive concepts such as curved spaces and parallel lines that behave differently than those in Euclidean space;
- Connect non-Euclidean principles to real-world applications;
- Strengthen creative problem-solving skills by tackling challenging and unfamiliar geometric situations that defy standard Euclidean intuition.

- Gain an understanding of key concepts for advanced studies in areas, such as topology, differential geometry, and physics;
- Develop an appreciation for the structuring and insightful power of mathematical thinking, recognizing that different assumptions can lead to equally valid, but distinct, mathematical structures;
- Recognize the historical and philosophical evolution of geometry, learn about the contributions to NEG of mathematicians like Gauss, Bolyai, and Lobachevsky, and understand the impact of non-Euclidean geometry on scientific thought.

Prerequisites: The prerequisites to be observed in this module are (1) familiarity with Euclidean Geometry, including its axioms and concepts such as parallel lines, triangles, and angle sums since Non-Euclidean Geometry builds upon and contrasts with these foundational principles; (2) experience (some) in mathematical proof methods, including direct proofs, proofs of contradiction, and inductive reasoning; (3) understanding of coordinate systems and algebraic representations of geometric objects, as these tools are frequently employed in analyzing hyperbolic and elliptic models; (4) basic knowledge of trigonometric functions and identities.

4. Theoretical foundation

Definition 9.1. An axiomatic system is a logical framework that starts with a set of postulates, statements assumed to be true without proof, which serves as the starting point from which, other statements (theorems) can be logically derived.

Euclidean geometry is based on five fundamental postulates that serve as its foundation. All theorems within the system are derived from them through logical deduction. These postulates provide the basis from which the entire structure of Euclidean geometry is built.

• The five Euclidean Postulates

1. **Postulate of Straight Lines**: It is possible to draw a straight line through any two points.
2. **Postulate of Extension of Lines**: Any straight line segment can be extended indefinitely in both directions.
3. **Postulate of Circles**: It is possible to draw a circle with any given center and radius.
4. **Postulate of Equality of Angles**: All right angles are equal to each other.
5. **V Postulate**: If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which the angles are less than two right angles.

Focusing on these five postulates, we notice that the fifth postulate differs in nature from the others. It is more complex and does not have the same intuitive level as the other four. This postulate admits an equivalent reformulation, from which the fifth postulate came to be called the parallel postulate, and which we state below. This is the reason why "Euclid's fifth postulate," and "Parallel postulate," are often used interchangeably.

6. **Parallel Postulate:** Given a line and a point not on it, there exists one, and only one line, parallel to the given line through the given point.

For many, many years, mathematicians tried to derive Euclid's fifth postulate from the first four, often by adding an "obvious" assumption that ultimately proved equivalent to the fifth postulate. Notable attempts include Proclus' straight-line distance assumption, Wallis' theory of similar triangles, Saccheri's quadrilateral examination, and Lambert's and Klügel's analyses of non-Euclidean possibilities.

In the 19th century, a breakthrough occurred, instead of finding contradictions in the denial of the fifth postulate, mathematicians accepted an alternative way of thinking, stating that through a given point, **more than one line** could be parallel to a given line. This shift in thinking led to hyperbolic geometry, which displayed surprising properties such as curved lines that remain equidistant and triangles with internal angle sums less than 180 degrees, proving to be a coherent alternative to Euclidean geometry.

Gauss, Lobachevsky, and Bolyai each established an axiomatic foundation for this new geometry. Gauss, however, opted not to publish his findings, anticipating significant controversy in academic circles. Encouraged by his father, János Bolyai published his work on hyperbolic geometry in 1832, following Lobachevsky's earlier publication in 1829. Although these pioneers did not provide mathematical proof of their system's consistency, they remained confident in its coherence and reliability.

• Models for the hyperbolic plane

We will now explore some of the most used models of planar hyperbolic geometry, highlighting the primitive hyperbolic terms and assigning interpretations to them within the context of Euclidean Geometry.

• *The Poincaré disc model*

Table 9.1 presents the Poincaré disk model, highlighting the primitive hyperbolic terms and the corresponding interpretation.

Hyperbolic Term	Interpretation
Point	Point interior to a given Euclidean circle $\mathcal{C}$
Line	A diameter of $\mathcal{C}$ or an arc of a circle that is orthogonal to $\mathcal{C}$
Plane	Interior of $\mathcal{C}$

Table 9.1. The Poincaré Disc Model

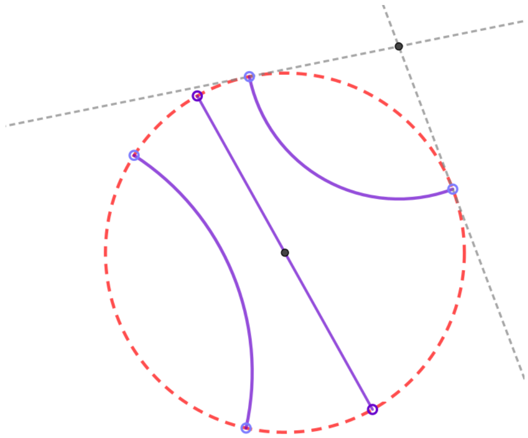

Figure 9.1. The Poincaré Disc Model.

The Poincaré disk model has historical significance as it was crucial in showing the relative consistency of hyperbolic geometry relative to Euclidean geometry.

In this model, the measure of an angle, determined in Euclidean terms, directly corresponds to the measure of a hyperbolic angle. However, the relationship between hyperbolic and Euclidean distances is much more complex. Essentially, in the hyperbolic case, distance scales are not uniform appearing to be bigger as they approach the border of C, reflecting the nature of hyperbolic space.

- *The Klein model*

The Klein model is very similar to the Poincaré disk model, see Table 9.2, using a more easily visualized interpretation of the primitive term line, however, in the Klein model, neither the angular nor the distance measurements coincide with the corresponding Euclidean measurements.

Hyperbolic Term	Interpretation
Point	Point interior to a given Euclidean circle $\mathcal{C}$
Line	An open chord of $\mathcal{C}$
Plane	Interior of $\mathcal{C}$

Table 9.2: The Klein model

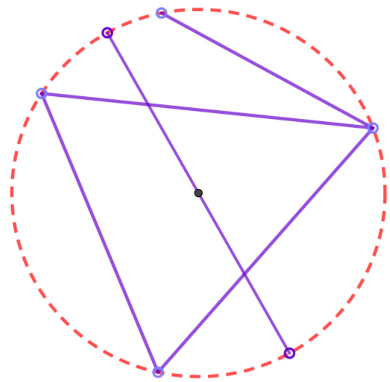

Figure 9.2: The Klein model

- *The upper half-plane model*

The upper half-plane model, like the Poincaré disk model, is of great historical importance in the development of hyperbolic geometry, bridging various branches of mathematics, and establishing the legitimacy of non-Euclidean geometries. In this model, points are represented by points located in the upper half of the Cartesian plane, and lines are represented as vertical rays and semi-circles orthogonal to the boundary line (see Table 9.3).

Hyperbolic Term	Interpretation
Point	Point in $\mathbb{H} = (x, y) \in \mathbb{R}^2\colon\ y > 0.$
Line	A subset of $\mathbb{H}$ of the form ${}_aL$ or ${}_cL_r$, where
	${}_aL = (x, y) \in \mathbb{H}\colon\ x = a,\ a \in \mathbb{R}$ and
	${}_cL_r = (x, y) \in \mathbb{H}\colon\ (x - C)^2 + y^2 = r^2,\ c \in \mathbb{R}, a \in \mathbb{R}^+$
Plane	$\mathbb{H}$

Table 9.3: The Upper half-plane model

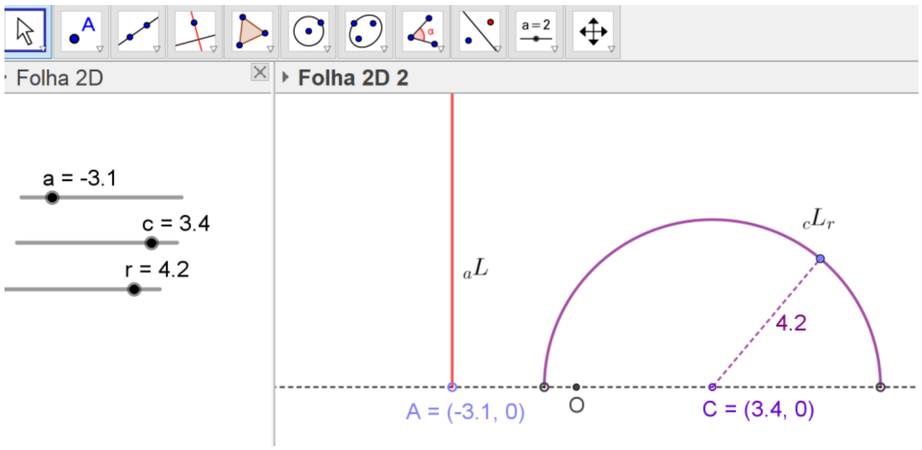

Figure 9.3. The Upper half-plane model

In this model, angles, measured using Euclidean concepts, are preserved as hyperbolic angles. However, the correspondence between Euclidean and hyperbolic distances is also non-trivial. Distances in the upper half-plane exhibit a scaling effect, appearing to stretch infinitely as they approach the horizontal limit, which serves as the "line at infinity". This catches the non-uniform nature of distance in hyperbolic space, as in the other models discussed before.

It is important to recognize that, despite their differences, all hyperbolic geometry models are isomorphic.

Theorem 9.1. *All models of hyperbolic planar geometry are isomorphic.*

Henceforth, we will work within the framework of the Poincaré half-plane model.

Definition 9.1. **Asymptotically parallel lines** (or limiting parallel lines) are lines that approach each other infinitely closely but never intersect, meeting at a boundary point at infinity. **Ultraparallel lines** are two lines that do not intersect and are not asymptotically parallel.

In Euclidean geometry, we are familiar with the concept of distance between two points, defined by the length of the straight line joining them. However, when we step into the domain of hyperbolic geometry, this key concept undergoes significant changes. Here, distances are calculated within a framework that respects the properties of hyperbolic space challenging our intuition.

The hyperbolic plane offers a rich structure in which distance must align with its postulates. In this setting, as we have already mentioned, see Table 3, a "straight line" does not always appear straight, and "traditional measurement" formulas do not incorporate the intrinsic curvature of the hyperbolic space.

The introduction of the hyperbolic distance allows us to look at new geometric relationships and explore intriguing results that differ from those we know in Euclidean geometry.

Definition 9.1. The **hyperbolic distance**, d_H, between the hyperbolic points P and Q is given by,

$$d_H(P,Q) = \begin{cases} \left| ln \dfrac{\overline{PA}}{\overline{QA}} \dfrac{\overline{PB}}{\overline{QB}} \right| & \text{if } \; P,Q \in_r L_c \\ \left| ln \dfrac{\overline{PC}}{\overline{QC}} \right| & \text{if } \; P,Q \in_a L \end{cases}$$

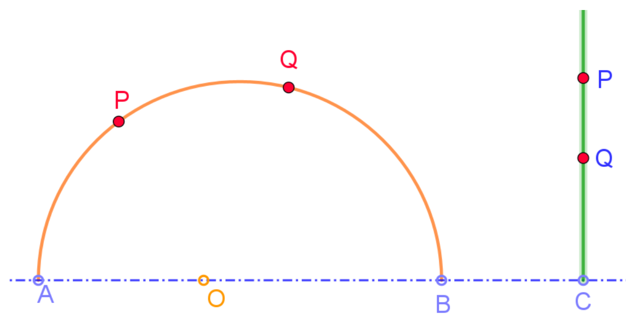

Figure 9.4.

where $\overline{PA}$ stands for the Euclidean distance between P and A, and ln denotes the natural logarithm.

The **hyperbolic angle** produced by the hyperbolic segments connecting points P to Q and P to R, respectively, is the angle formed by the tangents to these segments.

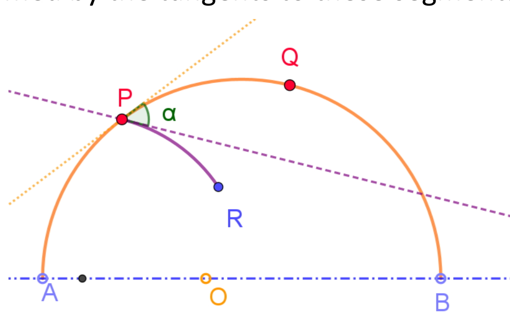

Figure 9.5.

As mentioned before, a great advantage of the upper half-plane model is its conformality. The measure of a hyperbolic angle is precisely the Euclidean angular measure of the angle formed by two curves.

In Euclidean geometry, the sum of the interior angles of a triangle is always 180^o, so the defect is always zero, but in hyperbolic geometry, this is not the case.

Theorem 9.2. *In hyperbolic geometry, the sum of the interior angles of any triangle is always less than* π *radians* (180^0).

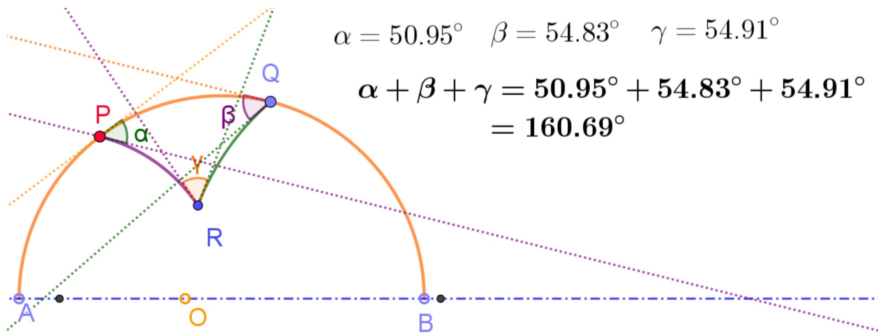

Figure 9.6.

Definition 9.3. In hyperbolic geometry, a triangle with one or more vertices on the boundary at infinity is known as an **asymptotic triangle**. If a triangle has one, two, or three vertices at infinity, it is referred to as a **singly asymptotic**, **doubly asymptotic**, or **triply asymptotic** triangle, respectively.

Theorem 9.3. *Let* $[ABC]$ *be a triangle in* $\mathbb{H}$ *with angle measures* α, β, *and* γ. *Then the area of* $[ABC]$ *is* $|[ABC]| = \pi - (\alpha + \beta + \gamma)$.

Definition 9.4. The **defect** of a triangle is defined as the difference between 180^o and the sum of the interior angles of the triangle.

A Saccheri quadrilateral is a special type of quadrilateral used to explore properties of both Euclidean and non-Euclidean geometries. It is named after Giovanni Girolamo Saccheri, an Italian mathematician who studied these figures extensively in his efforts to prove Euclid's parallel postulate.

Definition 9.5. A Saccheri quadrilateral is a four-sided figure (quadrilateral) in which, two opposite sides (called the legs) are equal in length and perpendicular to a third side (called the base). The fourth side is called the summit and is generally not equal in length to the base.

Theorem 9.4. *Comparing behaviors between Euclidean and hyperbolic geometries we can conclude that:*

	Euclidean	***Hyperbolic***
Two distinct lines intersect in	*at most one point*	*at most one point*
Given a line m and point P there exists	*exactly one*	*at least*
Nonintersecting lines	*are equidistant*	*are **never** equidistant*
The summit angles in a Saccheri quadrilateral are	*right*	*acute*
Two distinct lines perpendicular to the same line are	*parallel*	*ultraparallel*
The angle sum of a triangle is	*equal to 180^o*	*less than 180^o*
The area of a triangle is	*independent of its angle sum*	*proportional to the defect*
Two triangles with congruent corresponding angles are	*similar*	*congruent*

Table 9.4: Comparative results - Adapted from the Table 2.1 of [J. C] pp. 71

In the following table, we present a comparison, adapted from the table presented in [Judith Cedberg], between Euclidean geometry and hyperbolic geometry.

5. Applications to everyday life

Hyperbolic geometry has a surprising number of applications in everyday life, particularly in fields that require modeling complex or curved spaces.

The internet and other networks (like social networks) can be modeled using hyperbolic geometry, where distances help in understanding connectivity and efficiency of data routing, see [Boguná et al.]. Systems that account for Earth's curvature as GPS, use hyperbolic geometry principles to improve accuracy, especially in long-distance navigation over large regions, [Jekeli]. In projects involving cellular networks, hyperbolic geometry offers a powerful framework for interpreting and optimizing network properties, from resilience to navigability, providing both theoretical and practical value in understanding and managing complex systems [Faqeeh et al].

Hyperbolic geometry appears in the design of certain architectural structures and artworks. As stated in [Gawell], "The use of hyperbolic geometry in architecture can be traced through the work of prominent engineers, designers, and architects of the twentieth century, including P.L. Nervi, M. Nowicki, E. Saarinen, O. Niemeyer, F. Candela, E. Torroja. The work of engineers is particularly interesting, as it shows their search for the optimum structural forms using hyperbolic geometry".

The Hyperbolic Embroidery Project (Loom Hyperbolic, 2012), held in Marrakech, was inspired by Moroccan traditional craftsmanship, particularly the art of weaving cotton on a stable wooden frame, [Dumitrascu].

Figure 9.7.

In virtual reality (VR) and gaming, hyperbolic geometry opens new creative possibilities by allowing developers to design immersive, non-Euclidean spaces that defy conventional rules, enhancing engagement and exploration. In the medical field, hyperbolic geometry significantly aids in the interpretation of complex imaging data, such as MRI and CT scans. By mapping structures like the brain's convoluted surface in hyperbolic terms, medical professionals gain a more accurate understanding of intricate anatomical features, improving diagnostics and analysis of conditions that affect these areas. Hyperbolic models help optimize traffic flow and urban planning by accurately simulating and analyzing movement patterns over large, densely populated areas. This helps city planners make informed decisions to enhance the efficiency of road networks and pedestrian systems in sprawling urban environments.

In computer vision, hyperbolic geometry enhances image recognition by allowing algorithms to interpret and analyze curved forms and non-Euclidean spatial relationships, which is crucial for applications in autonomous systems and object detection.

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10. TOPIC: Measurements and units in astronomy

1. Justification for topic choice

The following aspects make the topic a valuable and appealing addition to bachelor-level education, equipping students with practical knowledge, interdisciplinary skills, and a deeper understanding of both historical and modern scientific methods.

- ### Interdisciplinary Learning

This topic integrates geometry, astronomy, and technology, offering students a hands-on understanding of how mathematical concepts apply to real-world phenomena. It bridges the gap between theoretical mathematics and practical applications in astronomy, which is valuable for students pursuing careers in STEM fields.

- ### Relevance to Modern Technologies

With advancements in technology, particularly in virtual reality (VR), students gain exposure to cutting-edge tools. The use of VR glasses provides an immersive learning experience, helping students visualize and interact with astronomical phenomena that are otherwise abstract or difficult to observe directly. This prepares students for modern data visualization and technological tools widely used in scientific research and industry.

- ### Historical and Practical Significance

Understanding the historical role of geometry in astronomy helps students appreciate the development of scientific thought and methods. It also provides a solid foundation for modern astronomical measurements, such as calculating distances between celestial bodies and understanding planetary motion, which are still based on geometric principles.

- ### Skill Development

The course helps students develop problem-solving and analytical skills by applying geometric methods to astronomical problems. These skills are transferable to various fields, such as physics, engineering, and data science. Additionally, using VR tools enhances digital literacy and spatial reasoning, important competencies in today's job market.

- ### Appeal to Curiosity and Exploration

Astronomy naturally sparks curiosity, making the subject engaging for students. By combining it with geometry and VR, students can explore vast concepts like the size of the universe, planetary motion, and celestial events, in a tangible, interactive way. This increases engagement and motivation, fostering a deeper interest in both mathematics and science.

2. Historical background

Ancient Civilizations and Early Geometry in Astronomy. Babylonians and Egyptians: Early civilizations like the Babylonians and Egyptians made foundational contributions to astronomy. They used simple geometric methods to track celestial objects, helping them predict eclipses, solstices, and equinoxes, essential for agriculture and religious rituals.

Greek Astronomy. The Greeks were instrumental in advancing the use of geometry for astronomical purposes. Thales of Miletus and Pythagoras laid the groundwork, and Hipparchus applied geometric principles to create the first models of celestial motions.

Ptolemy's Almagest. In the 2nd century AD, Ptolemy developed a sophisticated geocentric model of the universe using complex geometric constructions, including circles and epicycles, to explain planetary motion.

Islamic Golden Age. Islamic Astronomers: During the Islamic Golden Age, scholars like Al-Battani and Ibn al-Haytham refined the use of geometry to improve astronomical calculations. They preserved Greek works, improved observational techniques, and used geometric models to predict celestial events with greater accuracy. The Renaissance and the Copernican Revolution Nicolaus Copernicus: In the 16th century, Copernicus challenged the geocentric model with his heliocentric theory, using geometry to describe the orbits of planets around the Sun.

Johannes Kepler. Kepler's laws of planetary motion, derived using geometric reasoning, revolutionized our understanding of planetary orbits, replacing circular paths with ellipses.

Galileo Galilei. Galileo's use of the telescope for astronomical observations further solidified the role of geometry in interpreting celestial phenomena.

Modern Developments. Isaac Newton: Newton's laws of motion and universal gravitation, coupled with his use of geometry and calculus, provided a comprehensive mathematical framework for understanding celestial mechanics.

The 20th Century and General Relativity. Einstein's theory of general relativity used advanced geometry (non-Euclidean) to describe the curvature of spacetime around massive objects, fundamentally changing our understanding of gravity and astronomical phenomena.

This historical progression highlights how geometry has been a critical tool in the development of astronomical measurement techniques, setting the stage for modern applications, such as virtual reality-based simulations.

3. Learning outcomes

This modulus not only focuses on enhancing the students' knowledge of geometry and astronomy but also aims to develop a variety of skills - technical, analytical, and interdisciplinary - that will benefit them in their academic and professional careers.

1. Understanding of Geometrical Principles in Astronomy: Students will be able to understand and apply fundamental geometric concepts used in astronomical

measurements, such as angular distance, parallax, triangulation, and spherical geometry. These principles are the foundation of measuring distances, angles, and positions of celestial bodies, critical for both historical and modern astronomy.

2. Application of Geometry to Real-World Problems: Students will be motivated to use geometry to solve practical problems in astronomy, such as calculating distances between celestial bodies, determining the size of planets, and predicting planetary motion. This outcome ensures that students can take theoretical geometric principles and apply them in concrete astronomical contexts, fostering problem-solving skills.
3. Familiarity with Historical and Modern Techniques: Students will gain knowledge of both historical methods (e.g., Eratosthenes' measurement of the Earth) and modern technologies (e.g., parallax measurements with satellites) in astronomical measurements. Understanding the evolution of measurement techniques shows students the progress of scientific thought and enhances their grasp of the current state of the field.
4. Use of Technological Tools for Visualization: Students will be able to effectively use virtual reality (VR) technology and other digital tools to visualize and interact with geometric representations of astronomical measurements. This provides students with valuable technical skills in working with advanced technologies, which are becoming increasingly important in scientific research and education.
5. Improvement of Spatial and Analytical Thinking: Students will enhance their spatial reasoning and analytical thinking by working with three-dimensional models of the universe and solving geometric problems in astronomy. The ability to think spatially and analytically is crucial in many STEM disciplines, and these skills are sharpened through geometry-based astronomical measurements.
6. Development of Interdisciplinary Knowledge: Students will understand how geometry is interconnected with other fields, such as physics, astronomy, and engineering, and will be able to integrate knowledge from multiple disciplines to solve complex problems. This encourages an interdisciplinary approach to learning, showing students how mathematical concepts can be applied across different scientific areas.
7. Critical Thinking and Scientific Inquiry: Students will develop critical thinking skills by analyzing data from astronomical measurements and questioning the assumptions and limitations of different geometric models used in astronomy. Encouraging students to think critically about the methods they use fosters deeper scientific inquiry and helps them to approach problems from a more rigorous perspective.
8. Historical Appreciation of Scientific Development: Students will gain an appreciation for the historical contributions of mathematicians and astronomers to the field of geometry and its role in advancing our understanding of the universe.

   Justification: This fosters a broader appreciation for the historical context in which scientific knowledge develops, highlighting the continuous nature of discovery.

9. Enhanced Engagement with Abstract Concepts: By using VR technology and geometric tools, students will be able to engage more deeply with abstract astronomical concepts and improve their ability to visualize complex systems. Interactive tools make abstract concepts more accessible and engaging, improving students' ability to retain and understand difficult material.

Prerequisites:

4. Theoretical foundations

- ### Euclidean Geometry: Basic notions.

Euclidean Geometry is based on the postulates of the Greek mathematician Euclid, as outlined in his work Elements. It studies the properties and relationships of points, lines, angles, surfaces, and solids in a two-dimensional or three-dimensional space.

The basic notions of Euclidean Geometry include the following:

1. Points and Lines:

   In classical Euclidean geometry, a point is a primitive notion, defined as "that which has no part". A point is an has no dimensions, only position. A line: Euclid defined a line as an interval between two points and claimed it could be extended indefinitely in either direction.

2. Distance between two points: The distance between two points, $A(x_A, y_A)$ and $B(x_B, y_B)$ in a plane can be calculated using the distance formula:

   $$d(A, B) = \sqrt{(x_A - x_b)^2 + (y_A - y_B)^2}.$$

   In three-dimensional space, the distance between $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ is calculated as

   $$d(A, B) = \sqrt{(x_A - x_b)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

3. Angles:
   An angle is formed by two rays with a common endpoint. The sum of angles in a triangle is always 180°.
   The most common types of angles are right angles (90 degrees), acute angles (less than 90 degrees), and obtuse angles (greater than 90 degrees).

   Triangles and the Pythagorean Theorem: For a right triangle with legs a and b and hypotenuse c, the Pythagorean Theorem states: $a^2 + b^2 = c^2$.
   This relationship is fundamental in calculating distances and working with right-angled triangles.

4. Perimeter and Area:

The perimeter P of a polygon is the sum of the lengths of its sides. For a rectangle with length l and width w:

$$P = 2l + 2w.$$

The area of a polygon is the measure of the space it encloses. For a rectangle:

$$A = l \times w.$$

For a triangle with base b and height h:

$$A = \frac{1}{2} b \times h.$$

Circles: A circle is the set of all points in a plane that are equidistant from a given point (the center).
The circumference C of a circle with radius r is:

$$C = 2\pi r.$$

The area of the circle is:

$$A = \pi r^2$$

- The key principles of Euclidean Geometry

The key principles of Euclidean Geometry include the following postulates:

1. **First Postulate:** A straight line segment can be drawn joining any two points.
2. **Second Postulate:** A finite straight line can be extended indefinitely.
3. **Third Postulate:** A circle can be drawn with any center and radius.
4. **Forth Postulate:** All Right Angles are congruent.
5. **The Parallel Postulate:** If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two Right Angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Euclidean geometry has been historically significant in the field of astronomy, especially for making simple astronomical measurements. Two fundamental applications include calculating distances using parallax and determining the altitude of celestial objects above the horizon.

- Spherical Geometry

Description: Spherical geometry deals with the properties and relationships of points, lines, and angles on the surface of a sphere, which is crucial for modeling the celestial sphere and the Earth's curvature. Unlike Euclidean geometry, the shortest path between two points on a sphere is an arc of a great circle, not a straight line.

In mathematics, a spherical coordinate system is a coordinate system for three-dimensional space where the position of a given point in space is specified by three real numbers: the radial distance r along the radial line connecting the point to the fixed point of origin; the polar angle θ between the radial line and a given polar axis; and the azimuthal angle φ as the angle of rotation of the radial line around the polar axis. Once the radius is fixed, the three coordinates (r, θ, φ), known as a 3-tuple, provide a coordinate system on a sphere, typically called the spherical polar coordinates. The plane passing through the origin and perpendicular to the polar axis (where the polar angle is a right angle) is called the reference plane (sometimes the fundamental plane).

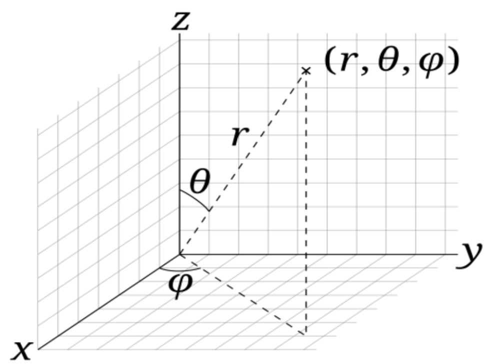

Figure 10.1.

To define a spherical coordinate system, one must designate an origin point in space, O, and two orthogonal directions: the zenith reference direction and the azimuth reference direction. These choices determine a reference plane that is typically defined as containing the point of origin and the x– and y–axes, either of which may be designated as the azimuth reference direction. The reference plane is perpendicular (orthogonal) to the zenith direction, and typically is designated *horizontal* to the zenith direction's *vertical*. The spherical coordinates of a point P then are defined as follows:

- The radius or radial distance is the Euclidean distance from the origin O to P.
- The inclination (or polar angle) is the signed angle from the zenith reference direction to the line segment $[OP]$. (Elevation may be used as the polar angle instead of inclination.)
- The azimuth (or azimuthal angle) is the signed angle measured from the azimuth reference direction to the orthogonal projection of the radial line segment $[OP]$ on the reference plane.

The sign of the azimuth is determined by designating the rotation that is the positive sense of turning about the zenith. This choice is arbitrary, and is part of the coordinate system definition. (If the inclination is either zero or 180° ($= \pi$ radians), the azimuth is arbitrary. If the radius is zero, both azimuth and inclination are arbitrary.)

The elevation is the signed angle from the xOy reference plane to the radial line segment $[OP]$, where positive angles are designated as upward, towards the zenith reference. Elevation is $90° = \frac{\pi}{2}$ radians minus inclination. Thus, if the inclination is $60° = \frac{\pi}{3}$ radians, then the elevation is $30° = \frac{\pi}{6}$ radians.

In linear algebra, the vector from the origin O to the point P is often called the position vector of P.

Any spherical coordinate triplet (or tuple) (r, θ, φ) specifies a single point of three-dimensional space. On the reverse view, any single point has infinitely many equivalent spherical coordinates. That is, the user can add or subtract any number of full turns to the angular measures without changing the angles themselves, and therefore without changing the point. It is convenient in many contexts to use negative radial distances, the convention being $(-r, \theta, \varphi)$, which is equivalent to $(r, \theta + 180°, \varphi)$ or $(r, 90° - \theta, \varphi + 180°)$ for any r, θ, and φ. Moreover, $(r, -\theta, \varphi)$ is equivalent to $(r, \theta, \varphi + 180°)$.

When necessary to define a unique set of spherical coordinates for each point, the user must restrict the range, aka interval, of each coordinate. A common choice is: radial distance: $r \geq 0$; polar angle: $0° \leq \theta \leq 180°$, or $0 \text{ rad} \leq \theta \leq \pi \text{ rad}$, and azimuth : $0° \leq \varphi < 360°$, or $0 \text{ rad} \leq \varphi < 2\pi \text{ rad}$.

But instead of the interval $[0°, 360°)$, the azimuth φ is typically restricted to the half-open interval $(-180°, +180°]$ or $(-\pi, +\pi]$ radians, which is the standard convention for geographic longitude.

For the polar angle θ, the range (interval) for inclination is $[0°, 180°]$, which is equivalent to elevation range (interval) $[-90°, +90°]$. In geography, the latitude is the elevation.

Even with these restrictions, if the polar angle (inclination) is $0°$ or $180°$, the elevation is $-90°$ or $+90°$, then the azimuth angle is arbitrary; and if $r = 0$, both azimuth and polar angles are arbitrary. To define the coordinates as unique, the user can assert the convention that (in these cases) the arbitrary coordinates are set to zero.

Just as the two-dimensional Cartesian coordinate system is useful - has a wide set of applications—on a planar surface, a two-dimensional spherical coordinate system is useful on the surface of a sphere. For example, one sphere that is described in Cartesian coordinates with the equation $x^2 + y^2 + z^2 = c^2$, with some $c > 0$, can be described in spherical coordinates by the simple equation $r = c$. (In this system, the sphere is adapted as a unit sphere, where the radius is set to unity and then can generally be ignored.)

This (unit sphere) simplification is also useful when dealing with objects such as *rotational matrices*. Spherical coordinates are also useful in analyzing systems that have some degree of symmetry about a point, including: volume integrals inside a sphere; the potential energy field surrounding a concentrated mass or charge; or global weather simulation in a planet's atmosphere.

The spherical coordinates of a point in the ISO convention (i.e. for physics: radius r, inclination θ, azimuth φ) can be obtained from its Cartesian coordinates (x, y, z) by the formulae:

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2}, \\
\theta &= \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} = \arccos\frac{z}{r} = \\
&\begin{cases}
\arctan\frac{\sqrt{x^2+y^2}}{z} & \text{if } z>0 \\
\pi+\arctan\frac{\sqrt{x^2+y^2}}{z} & \text{if } z<0 \\
+\frac{\pi}{2} & \text{if } z=0 \text{ and } \sqrt{x^2+y^2}\neq 0 \\
\text{undefined} & \text{if } x=y=z=0
\end{cases} \\
\varphi &= \operatorname{sgn}(y)\arccos\frac{x}{\sqrt{x^2+y^2}}
\begin{cases}
\arctan\frac{y}{x} & \text{if } x>0 \\
\arctan\frac{y}{x}+\pi & \text{if } x<0 \text{ and } y\geq 0 \\
\arctan\frac{y}{x}-\pi & \text{if } x<0 \text{ and } y<0 \\
+\frac{\pi}{2} & \text{if } x<0 \text{ and } y>0 \\
-\frac{\pi}{2} & \text{if } x=0 \text{ and } y<0 \\
\text{undefined} & \text{if } x=0 \text{ and } y=0
\end{cases}
\end{aligned}
$$

Conversely, the Cartesian coordinates may be retrieved from the spherical coordinates (radius r, inclination θ, azimuth φ), where $r>0$, $\theta\in[0,\pi]$, and $\varphi\in[0,2\pi)$ by

$$x = r\sin\theta\cos\varphi,$$
$$y = r\sin\theta\sin\varphi,$$
$$z = r\cos\theta.$$

- Distance Calculation

In spherical coordinates, given two points (r, θ, φ) and (r', θ', φ'), the distance between them is:

$$D = \sqrt{r^2 + r'^2 - 2rr'(\sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta')}.$$

Application: It is used in determining angular distances between stars, modeling planetary motion, and understanding concepts like declination and right ascension, essential for celestial navigation and positioning.

- Parallax

Parallax is the apparent shift in the position of an object when viewed from two different vantage points. In astronomy, it is used to measure the distance to nearby celestial objects, such as stars, by observing them from two locations on Earth or from different times in the Earth's orbit. Parallax is fundamental in measuring distances to nearby stars and planets, using simple geometric relationships between the Earth's movement and the apparent shift of celestial objects.

Let's say we observe a star from two different positions in the Earth's orbit (six months apart), and we measure the angle θ, which is the parallax angle. Using the basic principles of Euclidean geometry, we can form a right triangle, where:

- D is the distance from Earth to the star,
- d is the baseline (the distance between the two observation points, typically the diameter of Earth's orbit around the Sun), and
- θ is the parallax angle.

For small angles, the parallax distance formula is given by:

$$D \approx \frac{d}{2\tan(\theta)}.$$

- Lunar Parallax

The first parallax determination was for the Moon, by far the nearest celestial body. *Hipparchus*, the Greek astronomer (150 BCE), determined the Moon's parallax to be $58'$ for a distance of approximately 59 times Earth's equatorial radius, as compared with the modern value of $57'02.6''$, that is, a mean value of 60.2 times. Lunar parallax is directly determined from observations made at two places, such as G, Greenwich, England, and C, the Cape of Good Hope, South Africa, that are nearly on the same meridian. Two angles, z_1 and z_2, are observed, and other data are obtained from the latitudes of the observatories and the known size and shape of Earth. In practice, stars near the Moon are observed also to eliminate errors of refraction and instruments.

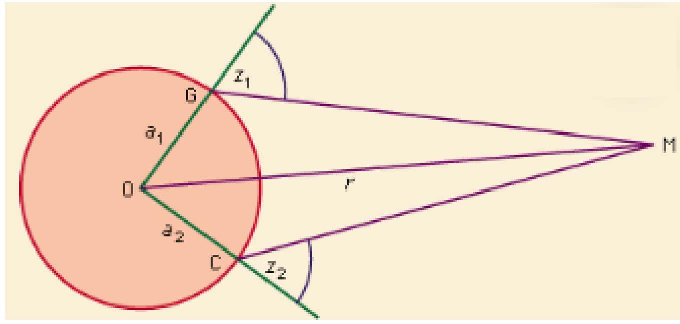

Figure 10.2. Measurement of parallax by observations from points G and C

Radar and laser measures of the distance from Earth to the Moon have provided a recent value of the lunar parallax. Radar and laser ranges have the advantage of being a direct distance measure, although the ranges are affected by variations in the surface topography of the Moon and require assumptions about the lunar radius and the centre of mass. The International Astronomical Union in 1964 adopted a value of $57'02.608''$ for the lunar parallax corresponding to a mean distance of 384,400 km (238,900 miles).

- ## Solar parallax

The basic method used for determining solar parallax is the determination of *trigonometric parallax*. In accordance with the law of gravitation, the relative distances of the planets from the Sun are known, and the distance of the Sun from Earth can be taken as the unit of length. The measurement of the distance or parallax of any planet will determine the value of this unit. The smaller the distance of the planet from Earth, the larger will be the parallactic displacements to be measured, with a corresponding increase in accuracy of the determined parallax. The most favourable conditions are therefore provided by the observation, near the time of opposition, of planets approaching close to Earth. The determination can be based either on simultaneous or nearly simultaneous observations from two different places on Earth's surface, or on observations made after sunset and before sunrise at the same place, when the displacement of the place of observation produced by the rotation of Earth provides the base line for the measurements.

The first reasonably accurate determination of the Sun's parallax was made in 1672 from observations of Mars at Cayenne, French Guiana, and Paris, from which a value of $9.5''$ was obtained.

Methods depending on velocity of light are also employed to ascertain solar parallax. The value of the velocity of light has been determined with very high precision and may be utilized in several different ways. A direct method is the converse of the procedure of a Danish astronomer Ole Rømer in the discovery of the velocity of light; i.e., to use the light equation, or time taken by the light to reach us at the varying distances of Jupiter, but great accuracy is hardly obtainable in this way. A second method is by means of the constant of aberration, which gives the ratio of the velocity of Earth in its orbit to the velocity of light. As aberration produces an annual term of amplitude $20.496''$ in the positions of all stars, its amount has been determined in numerous ways. Observations made at Greenwich in the years 1911 to 1936 gave the value $20.489'' \pm 0.003''$ leading to the value $8.797'' \pm 0.013''$ for solar parallax. This method is not free from the suspicion of systematic error.

The velocities of stars toward or away from Earth are determined from spectroscopic observations. By choosing times when the orbital motion of Earth is carrying it toward or from a star, astronomers are able to determine mathematically the velocity of the Earth in its orbit. In this way the solar parallax was found from observations at the Cape of Good Hope to be $8.802'' \pm 0.004''$.

Radar measures of the distance from Earth to Venus have provided the best determination of the solar parallax. By measuring the flight time of a radar pulse to Venus, the distance between the two planets can be obtained, allowing the determination of the unit distance between Earth and the Sun.

The present value for the astronomical unit is 149,597,871 km (92,955,807 miles). The principal limitations of using radar to measure the astronomical unit are the dependence on knowledge of the planetary orbits, the uncertainty in the value of the velocity of light, and the possibility of electromagnetic effects in the Earth–Venus plasma delaying the radar pulse.

- ## Stellar parallax

The stars are too distant for any difference of position to be perceptible from two places on Earth's surface, but, as Earth revolves at 149,600,000 km from the Sun, stars are seen from widely different viewpoints during the year. The effect on their positions is called annual parallax, defined as the difference in position of a star as seen from Earth and from the Sun. Its amount and direction vary with the time of year, and its maximum is $\frac{a}{r}$, where a is the radius of the Earth's orbit and r the distance of the star. The quantity is very small and never reaches 1/206,265 in radians, or $1''$ in sexagesimal measure.

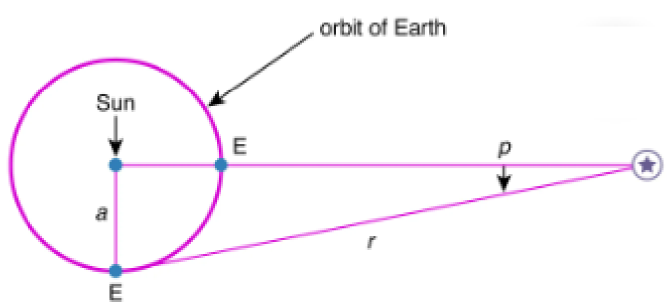

Figure 10.3. Stellar parallax

Using a heliometer designed by German physicist Joseph von Fraunhofer, German astronomer Friedrich Wilhelm Bessel was the first to measure stellar parallax in 1838. Choosing 61 Cygni, a star barely visible to the naked eye and known to possess a relatively high velocity in the plane of the sky, Bessel showed in 1838 that, after correcting for velocity, the star apparently moved in an ellipse every year. This back-and-forth motion was the annual parallax. Astronomers had known for centuries that such an effect must occur, but Bessel was the first to demonstrate it accurately. Bessel's parallax of about one-third of a second of arc corresponds to a distance of about 10.3 light-years from Earth to 61 Cygni, though Bessel did not express it this way. (The nearest star known is Alpha Centauri, 4.3 light-years away, with a parallax of about $0.75''$.)

- ## Kepler's Laws of Planetary Motion

In astronomy, Kepler's laws of planetary motion, published by Johannes Kepler absent the third law in 1609 and fully in 1619, describe the orbits of planets around the Sun. These laws replaced circular orbits and epicycles in the heliocentric theory of Nicolaus Copernicus with elliptical orbits and explained how planetary velocities vary. Kepler's three laws describe the motion of planets around the Sun using geometric principles.

The laws explain how planets trace elliptical orbits, how they sweep out equal areas in equal times, and the relationship between a planet's orbital period and its distance from the Sun. The three laws state that:

1. **Kepler's First Law: the Law of Ellipses.** Each planet's orbit about the Sun is an ellipse. The Sun's center is always located at one focus of the orbital ellipse. The Sun is at one focus. The planet follows the ellipse in its orbit, meaning that the planet to Sun distance is constantly changing as the planet goes around its orbit.] Kepler's First Law: each planet's orbit about the Sun is an ellipse. The Sun's center is always located at one focus of the orbital ellipse. The Sun is at one focus. The planet follows the ellipse in its orbit, meaning that the planet to Sun distance is constantly changing as the planet goes around its orbit.

2. **Kepler's Second Law: the Law of Equal Areas.** The imaginary line joining a planet and the Sun sweeps equal areas of space during equal time intervals as the planet orbits. Basically, that planets do not move with constant speed along their orbits. Rather, their speed varies so that the line joining the centers of the Sun and the planet sweeps out equal parts of an area in equal times. The point of nearest approach of the planet to the Sun is termed perihelion. The point of greatest separation is aphelion, hence by Kepler's Second Law, a planet is moving fastest when it is at perihelion and slowest at aphelion.

3. **Kepler's Third Law: the Harmonic Law.** The squares of the orbital periods of the planets are directly proportional to the cubes of the semi-major axes of their orbits. Kepler's Third Law implies that the period for a planet to orbit the Sun increases rapidly with the radius of its orbit. Thus we find that Mercury, the innermost planet, takes only 88 days to orbit the Sun. The earth takes 365 days, while Saturn requires 10,759 days to do the

same. Though Kepler hadn't known about gravitation when he came up with his three laws, they were instrumental in Isaac Newton deriving his theory of universal gravitation, which explains the unknown force behind Kepler's Third Law. Kepler and his theories were crucial in the better understanding of our solar system dynamics and as a springboard to newer theories that more accurately approximate our planetary orbits.

The elliptical orbits of planets were indicated by calculations of the orbit of Mars. From this, Kepler inferred that other bodies in the Solar System, including those farther away from the Sun, also have elliptical orbits. The second law establishes that when a planet is closer to the Sun, it travels faster. The third law expresses that the farther a planet is from the Sun, the longer its orbital period.

Isaac Newton showed in 1687 that relationships like Kepler's would apply in the Solar System as a consequence of his own laws of motion and law of universal gravitation.

The usefulness of Kepler's laws extends to the motions of natural and artificial satellites, as well as to stellar systems and extrasolar planets. As formulated by Kepler, the laws do not, of course, take into account the gravitational interactions (as perturbing effects) of the various planets on each other. The general problem of accurately predicting the motions of more than two bodies under their mutual attractions is quite complicated; analytical solutions of the three-body problem are unobtainable except for some special cases. It may be noted that Kepler's laws apply not only to gravitational but also to all other inverse-square-law forces and, if due allowance is made for relativistic and quantum effects, to the electromagnetic forces within the atom.

Application: These laws help students understand how geometry is used to predict planetary orbits, calculate their positions at any time, and understand the geometric nature of the solar system's layout.

- ## The Celestial Sphere Model

Looking up at the sky and watching the Sun, the Moon, and the stars go by, it's easy to think that we are at the center of the universe, that everything revolves around our little world. Indeed, that is how most people throughout human history thought things were. They saw the Earth as the center of all things. Being the center of all creation made Earth a special place. The Sun, the Moon, and the five known planets all revolved around the world. Somewhere beyond Saturn, was the Firmament or Vault or Heaven where the stars resided. Some people saw the Firmament as a literal dome or sphere where the stars hung. They considered this celestial sphere to be a real, physical structure and all the stars were more or less the same distance from Earth.

Of course, today we know there is not physical celestial sphere and that the stars are much further away from us than ancient thought. In fact, they are not all equidistant from us. Stars that appear to be close together in the sky may in fact be hundreds or thousands of light-years away when we consider them in three dimensions. They only appear to be close together because they happen to be in roughly the same line of sight from our vantage point. Think of an optical illusion that makes two objects look close together even when they are in fact, far apart.

The celestial sphere is an imaginary sphere centered on Earth, onto which all celestial objects are projected. Understanding the geometry of the celestial sphere is fundamental for locating stars and planets using coordinates like declination and right ascension.

The horizontal coordinate system is a celestial coordinate system that uses the observer's local horizon as the fundamental plane to define two angles of a spherical coordinate system: altitude and azimuth. Therefore, the horizontal coordinate system is sometimes called the az/el system, the alt/az system, or the alt-azimuth system, among others. In an altazimuth mount of a telescope, the instrument's two axes follow altitude and azimuth.

The celestial coordinate system divides the sky into two hemispheres: the upper hemisphere, where objects are above the horizon and are visible, and the lower hemisphere, where objects are below the horizon and cannot be seen, since the Earth obstructs views of them.[a] The great circle separating the hemispheres is called the celestial horizon, which is defined as the great circle on the celestial sphere whose plane is normal to the local gravity vector. In practice, the horizon can be defined as the plane tangent to a quiet, liquid surface, such as a pool of mercury. The pole of the upper hemisphere is called the zenith. The pole of the lower hemisphere is called the nadir.

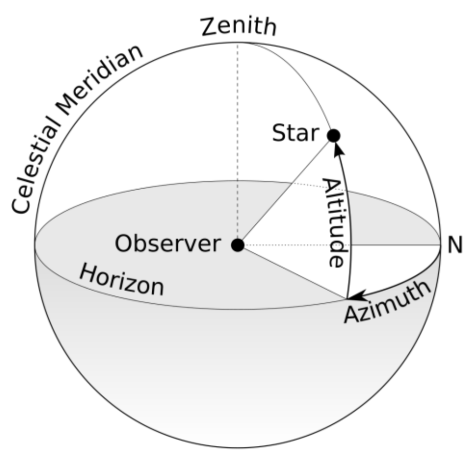

Figure 10.2. Celestial Sphere

The following are two independent horizontal angular coordinates:

Altitude (alt.), sometimes referred to as elevation (el.) or apparent height, is the angle between the object and the observer's local horizon. For visible objects, it is an angle between and Azimuth (az.) is the angle of the object around the horizon, usually measured from true north and increasing eastward.

A horizontal coordinate system should not be confused with a topocentric coordinate system. Horizontal coordinates define the observer's orientation, but not location of the origin, while topocentric coordinates define the origin location, on the Earth's surface, in contrast to a geocentric celestial system.

Application: The model helps describe how the positions of stars change throughout the night and year, and it is essential for comprehending the geometric principles underlying star charts, navigation, and astronomical observations.

These additional foundations round out the topic by introducing systems for positioning and locating celestial objects, as well as key geometric laws that are fundamental to understanding the behavior of light and paths on spherical surfaces, both of which are critical in astronomical measurements. These theoretical foundations provide the mathematical and geometrical frameworks necessary to understand and perform astronomical measurements, from the classical methods developed by the Greeks to modern relativistic corrections in celestial mechanics.

• Methodology of teaching astronomy with VR

Studying Euclidean geometry with VR glasses can provide a transformative, immersive learning experience that allows students to interact with geometric shapes and concepts in ways that go beyond traditional 2D diagrams and chalkboard representations. Here's how various aspects of Euclidean geometry can be explored and enhanced using VR:

1. **Visualizing and Interacting with 3D Shapes Exploring Geometric Solids:**

   In VR, students can manipulate and explore 3D versions of geometric solids such as cubes, spheres, pyramids, and prisms. They can view them from different angles, rotate them, and even "walk" around the shapes to understand their properties more deeply. VR allows students to dissect solids, revealing cross-sections to better understand relationships between 2D shapes and 3D objects, such as how slicing a cone can create circles, ellipses, or parabolas. Interactive Proofs:

   Rather than just drawing proofs on paper, students can use VR to interactively construct proofs. For example, proving the Pythagorean theorem can be done by manipulating squares on the sides of a right triangle and physically "rearranging" them in a way that demonstrates the relationship between the areas. With interactive transformations like rotations, translations, and reflections, students can directly manipulate figures to understand congruence, similarity, and symmetry.

2. **Exploring Euclid's Postulates in 3D:**

   In VR, students can directly draw a straight line between any two points in 3D space, exploring the concept of straight lines not just on a plane but within a larger 3D environment.

   By interacting with lines, students can see line extensions in real time, understanding how lines behave in Euclidean space and how this postulate works in both 2D and 3D settings.

Students can easily draw circles by choosing any center point and radius, then view them from different angles and even observe them as cross-sections of spheres or other solids, reinforcing their understanding of circular symmetry. Parallel Postulate (Given a line and a point not on the line, only one line parallel to the given line can be drawn through the point):

VR can visually demonstrate parallel lines in both two-dimensional planes and three-dimensional space, showing that parallel lines do not meet, no matter how far they are extended, and students can explore the concept by drawing lines themselves.

3. **Understanding Angle and Distance Relationships Measuring Angles:**

Students can use virtual tools to measure angles between intersecting lines, polygons, or solid objects, gaining a clear sense of how angles work in 3D space. They can interact with dynamic angle manipulation, allowing them to adjust angles and immediately observe how other geometric properties change as a result (such as seeing how the sum of interior angles in a triangle always equals 180 degrees, even as the triangle's shape changes). Distance and Area Calculations: VR environments can provide interactive tools to measure distances between points or along curves, reinforcing the concept of line segments and distances. Students can also calculate areas and volumes by selecting faces of polygons and polyhedra, providing a hands-on way to internalize the formulas for area and volume.

4. **Transformations and Symmetry Translations, Rotations, and Reflections:**

VR makes it easy to translate, rotate, and reflect objects in space. For example, students can move a triangle across a plane or rotate a cube in space to explore the effects of these transformations on different geometric properties (such as congruence or symmetry). They can also study symmetry by observing reflections in various planes, and understand how symmetry groups work by interacting with objects like regular polyhedra (e.g., platonic solids). Tessellations and Tilings:

Students can create tessellations by repeating polygons in space, helping them understand periodic patterns and how regular polygons can tile a plane. They can view these from different perspectives, reinforcing spatial visualization.

5. **Dynamic Construction and Experimentation Constructing Geometric Figures:**

Using VR tools, students can construct basic geometric shapes such as triangles, quadrilaterals, circles, and polygons in both 2D and 3D space. They can manipulate these constructions, changing side lengths, angles, and orientations while observing how properties like parallelism or congruence are maintained. Interactive Geometry Problems: VR offers a platform for solving geometry problems dynamically. For instance, constructing an equilateral triangle or bisecting an angle can become an interactive challenge where students manipulate virtual compasses and straightedges, as they would in traditional compass-and-ruler constructions, but with the added benefit of immersive visualization.

Application: This framework underpins the simplest astronomical measurements, such as calculating distances using parallax or determining the altitude of celestial objects above the horizon.

5. Applications to everyday life

- ### Space Exploration and Satellite Navigation.

Application: Geometry plays a critical role in calculating trajectories for space missions, whether for launching satellites, sending rovers to Mars, or exploring distant planets. For instance, the orbits of satellites are determined using geometric principles of conic sections (ellipses, parabolas), and precise measurements ensure the correct positioning of spacecraft.

Relevance: Understanding these geometric foundations allows engineers to predict and control spacecraft trajectories, ensure satellite coverage for GPS systems, and avoid collisions with space debris.

- ### Global Positioning Systems (GPS).

Application: GPS technology relies heavily on geometric triangulation to calculate precise locations on Earth. By using signals from multiple satellites, which are positioned according to geometric principles, the system can pinpoint an object's location with high accuracy.

Relevance: The application of geometry in GPS is crucial for navigation in everyday life, from smartphones to transportation systems, and for more advanced uses such as autonomous vehicles, mapping, and geospatial analysis.

- ### Astronomical Observations and Research.

Application: Modern astronomical observatories and space telescopes, such as the Hubble Space Telescope or the upcoming James Webb Space Telescope, use geometry to calculate distances between celestial objects, measure star positions, and understand galactic structures. Techniques like astrometry (the precise measurement of star positions) and parallax rely on geometric principles.

Relevance: These measurements are essential for scientific discoveries, such as identifying exoplanets, studying dark matter, and mapping the structure of the universe.

- ### Stellar Parallax and Distance Measurement.

Application: Stellar parallax, the apparent shift in position of a nearby star against the background of distant stars as Earth orbits the Sun, is used to measure distances to stars within our galaxy. This technique employs simple geometric principles of triangles and angles.

Relevance: Parallax remains a fundamental tool for determining stellar distances, which is crucial for understanding the scale of the universe, star formation, and galaxy structure.

- ### Astronomical Imaging and 3D Visualization.

Application: Modern technologies, including VR (Virtual Reality) and 3D modeling, use geometry to create immersive simulations of celestial phenomena. VR glasses allow users to explore geometrical models of planetary systems, orbits, and stellar formations, providing a deeper understanding of spatial relationships in the cosmos.

Relevance: This is important for education, scientific outreach, and research, offering a more intuitive grasp of complex astronomical systems and enabling interactive exploration of the universe.

- ### Planetary Motion and Orbital Mechanics.

Application: Geometry is fundamental in calculating and predicting planetary orbits, based on Kepler's laws of planetary motion and Newtonian mechanics. These calculations are used to launch satellites, plan interplanetary missions, and predict celestial events like eclipses and transits.

Relevance: Accurate predictions of orbital paths are critical for everything from the daily operations of space agencies like NASA to commercial satellite services.

- ### Cosmology and the Shape of the Universe.

Application: In cosmology, geometry is applied to study the shape and structure of the universe. Theories like general relativity, which describe the curvature of spacetime due to gravity, use non-Euclidean geometry to explain large-scale structures, such as black holes, gravitational lensing, and the expansion of the universe.

Relevance: Understanding the geometry of spacetime helps scientists develop models for phenomena such as the Big Bang, dark energy, and the ultimate fate of the universe, providing a foundation for modern theoretical physics.

- ### Astrophotography and Time-Lapse Observation.

Application: Geometrical algorithms are used in astrophotography to capture and process images of celestial bodies over time. Time-lapse observation, in particular, requires precise geometric calculations to track the apparent movement of stars and planets across the sky.

Relevance: These techniques are vital for both amateur and professional astronomers who document celestial events like meteor showers, planetary conjunctions, and star trails.

These applications demonstrate how geometry continues to be a vital tool in both the theoretical and practical aspects of modern astronomy, from everyday technologies like GPS to advanced scientific endeavors such as space exploration and cosmology.

https://math.libretexts.org/Bookshelves/Geometry/Modern_Geometry_(Bishop)/03%3A_Introduction_to_Hyperbolic_Geometry/3.01%3A_Hyperbolic_Geometry

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